Comments on the estimate for Pareto Distribution

U. J. Dixit ¹
M. Jabbari Nooghabi ²
Department of Statistics, University of Mumbai, Mumbai-India

Abstract

Dixit and Jabbari Nooghabi (2010) had derived the MLE and UMVUE of the probability density function (pdf) and cumulative distributive function (cdf). Further, it had been shown that MLE is more efficient than UMVUE. He, Zhou and Zhang (2014) have also derived the same and made a remark that the work of Dixit and Jabbari Nooghabi (2010) is not correct. We have made a comments with detail algebra that our results are correct. Further, we have also given the R code.

Key Words: Pareto distribution, Maximum likelihood estimator, Uniform minimum variance unbiased estimator, Probability density function (pdf), Cumulative distribution function (cdf), Comments, R code.

1 Introduction

The Pareto distribution has been used in connection with studies of income, property values, insurance risk, migration, size of cities and firms, word frequencies, business mortality, service time in queuing systems, etc.

The objective of this paper is to discuss efficient estimation of pdf and CDF of Pareto distribution which has been one of the most distinguished candidates for the honor of explaining the distribution of incomes, assets, etc.

We assume that random variable X has Pareto distribution with parameter α and k (known) and its probability density function (pdf) is as,

$$f_X(x) = \frac{\alpha k^{\alpha}}{x^{\alpha+1}}, \qquad 0 < k \le x, \quad \alpha > 0$$

and distribution function

$$F_X(x) = 1 - \left(\frac{k}{x}\right)^{\alpha}, \quad k \le x$$

In economics, where this distribution is used as an income distribution, k is some minimum income with a known value. As abadi (1990) derived the uniformly minimum variance unbiased estimator (UMVUE) of the probability density function (pdf), the distribution function (cdf) and the r^{th} moment.

In this paper, we will give the detail algebra of Dixit and Jabbari Nooghabi's (2010) paper. Also, We have made a comments that our results are correct. Further, we have also given the R code.

Dixit and Jabbari Nooghabi (2010) had derived the MLE and UMVUE of the probability density function (pdf) and cumulative distributive function (cdf). Further, it had been shown that MLE is more efficient than UMVUE. He, Zhou and Zhang (2014) have also derived the same and made a remark that the work of Dixit and Jabbari Nooghabi (2010) is not correct.

We like to make some comments as follows.

- 1. We have verified our results and they are correct. We have given the detail algebra and R program. See the attachment.
- 2. Examples given by He et al. (2014) are not correct. One should note MSE of $\hat{f}(x)$ or $\hat{F}(x)$ is a function of parameters. By this one cannot prove anything. Only, one can calculate $\hat{f}(x)$ or $\hat{F}(x)$.

¹E-mail: ulhasdixit@yahoo.co.in

²E-mail: jabbarinm@yahoo.com, jabbarinm@um.ac.ir

- 3. According to definition of the modified Bessel function in Olver, Lozier, Boisvert, et al. (2010) in the Theorem 1. the notation $K_{(n-r)}(2\sqrt{nr\alpha z})$ should be $K_{(r-n)}(2\sqrt{nr\alpha z})$. Also, the notation $K_{(n)}(2\sqrt{nr\alpha z})$ should be $K_{(-n)}(2\sqrt{nr\alpha z})$.
- 4. After the Theorem 2. the Kummer confluent hypergeometric function is wrong and the correct version is

$$U(a,b,c) = \frac{1}{\Gamma(a)} = \int_0^\infty t^{a-1} (1+t)^{b-a-1} e^{-ct} dt.$$

2 Main Result

In this section, we give the detail algebra of the paper Dixit and Jabbari Nooghabi (2010). The details of finding result of the second chapter of that paper are as follows.

$$X_1, \dots, X_n \stackrel{iid}{\sim} f(x) = \frac{\alpha k^{\alpha}}{x^{\alpha+1}}, \quad 0 < k \le x, \ \alpha > 0$$
 (1)

$$L(x_1, ..., x_n, \alpha, k) = \frac{\alpha^n k^{n\alpha}}{\prod_{i=1}^n x_i^{\alpha+1}} \prod_{i=1}^n \mathbf{I}(x_i - k),$$

where I is the indicator function defined as

$$\mathbf{I}(y) = \begin{cases} 1 & y > 0, \\ 0 & otherwise. \end{cases}$$

$$\Rightarrow \ln L(\underline{x}, \alpha, k) = n \ln(\alpha) + n\alpha \ln(k) - (\alpha + 1) \ln(\sum_{i=1}^{n} x_i)$$

$$\Rightarrow \frac{\partial \ln L(\underline{x}, \alpha)}{\partial \alpha} = \frac{n}{\alpha} + n \ln(k) - \ln \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \tilde{\alpha} = MLE(\alpha) = \frac{n}{\sum_{i=1}^{n} \ln(\frac{x_i}{k})}, \quad \frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{-n}{\alpha^2} < 0$$

$$\Rightarrow MLE \ of \ f(x) = f(x, \tilde{\alpha}) = \frac{\tilde{\alpha}k^{\tilde{\alpha}}}{x^{\tilde{\alpha}+1}}$$

$$\Rightarrow \tilde{f}(x) = \frac{\tilde{\alpha}k^{\tilde{\alpha}}}{x^{\tilde{\alpha}+1}}, \quad \tilde{\alpha} > 0, \ 0 < k \le x,$$

$$\Rightarrow \tilde{F}(x) = 1 - \left(\frac{k}{x}\right)^{\tilde{\alpha}}, \quad \tilde{\alpha} > 0, \ 0 < k \le x.$$
(2)

Put:

$$y = \ln\left(\frac{x}{k}\right) \Rightarrow dy = \frac{1}{x}dx$$

$$\stackrel{x=ke^y}{\Longrightarrow} f_Y(y) = ke^y \frac{\alpha k^\alpha}{(ke^y)^{\alpha+1}} = \alpha e^{-\alpha y}, \quad y > 0, \quad x \ge k$$

$$\Rightarrow f_Y(y) = \alpha e^{-\alpha y}, \quad y > 0 \text{ or } Y \sim \Gamma(1, \frac{1}{\alpha})$$

$$\Rightarrow S = \sum_{i=1}^n Y_i \sim \Gamma(n, \frac{1}{\alpha}),$$

$$g_S(s) = \frac{\alpha^n s^{n-1}}{\Gamma(n)} \exp(-\alpha s), \quad s > 0.$$
(3)

Let $w = \tilde{\alpha}$

$$(1) \Rightarrow w = \frac{n}{s} \Rightarrow s = \frac{n}{w} \Rightarrow \frac{ds}{dw} = \frac{-n}{w^2}$$

$$\Rightarrow Jacobian = \left|\frac{-n}{w^2}\right| = \frac{n}{w^2} \Longrightarrow g(w) = \frac{n}{w^2} \frac{1}{\Gamma(n)(\frac{1}{\alpha})^n} \left(\frac{n}{w}\right)^{n-1} e^{-n\alpha/w}, \quad w > 0, \quad \alpha > 0$$

$$\Rightarrow g(w) = \frac{(\alpha n)^n}{\Gamma(n)(w^{n+1})} \exp\left\{-\frac{\alpha n}{w}\right\}, \quad w > 0. \tag{4}$$

$$E(\tilde{\alpha}) = E(W) = \int_0^\infty w g(w) dw = \int_0^\infty w \frac{(\alpha n)^n}{\Gamma(n)(w^{n+1})} \exp\left\{-\frac{\alpha n}{w}\right\} dw = \frac{(\alpha n)^n}{\Gamma(n)} \int_0^\infty w^{-n} e^{-\frac{\alpha n}{w}} dw.$$

Put
$$z = \frac{1}{w} \Rightarrow dz = -\frac{1}{w^2}dw$$
, so

$$E(W) = \frac{(\alpha n)^n}{\Gamma(n)} \int_0^\infty z^{n-2} e^{-\alpha n z} dz = \frac{(\alpha n)^n}{\Gamma(n)} \Gamma(n-1) (\frac{1}{\alpha n})^{n-1} \int_0^\infty \frac{z^{n-2} e^{-\alpha n z}}{\Gamma(n-1) (\frac{1}{\alpha n})^{n-1}} dz.$$

$$\Rightarrow E(\tilde{\alpha}) = \frac{\alpha n}{n-1}.$$

$$E(\tilde{\alpha}^2) = E(W^2) = \int_0^\infty w^2 g(w) dw = \int_0^\infty w^2 \frac{(\alpha n)^n}{\Gamma(n)(w^{n+1})} \exp\left\{-\frac{\alpha n}{w}\right\} dw = \frac{(\alpha n)^n}{\Gamma(n)} \int_0^\infty w^{-n+1} e^{-\frac{\alpha n}{w}} dw.$$

Same as the previous

$$\begin{split} E(W^2) &= \frac{(\alpha n)^n}{\Gamma(n)} \int_0^\infty z^{n-3} e^{-\alpha n z} dz = \frac{(\alpha n)^n}{\Gamma(n)} \Gamma(n-2) (\frac{1}{\alpha n})^{n-2} \int_0^\infty \frac{z^{n-3} e^{-\alpha n z}}{\Gamma(n-2) (\frac{1}{\alpha n})^{n-2}} dz. \\ \Rightarrow E(\tilde{\alpha}^2) &= \frac{(\alpha n)^2}{(n-1)(n-2)}. \end{split}$$

Therefore

$$MSE(W) = V(W) + (E(W) - \alpha)^{2} = E(W^{2}) - 2\alpha E(W) + \alpha^{2}$$

$$= \frac{(\alpha n)^{2}}{(n-1)(n-2)} - 2\alpha \frac{\alpha n}{n-1} + \alpha^{2}.$$

$$\Rightarrow MSE(\tilde{\alpha}) = MSE(W) = \frac{\alpha^{2}(n^{2} + n - 2)}{(n-1)^{2}(n-2)}.$$

Proof of the Theorem 1.:

(A)

$$E(\tilde{f}(x)) = \int \tilde{f}(x)g(w)dw = \int_0^\infty \frac{wk^w}{x^{w+1}} \frac{(\alpha n)^n}{\Gamma(n)w^{n+1}} e^{-\alpha n/w} dw$$
$$= \frac{(\alpha n)^n}{\Gamma(n)x} \int_0^\infty \left(\frac{k}{x}\right)^w \frac{e^{-\alpha n/w}}{w^n} dw.$$

Put $(\frac{k}{x})^w = e^{w \ln(\frac{k}{x})}$, then

$$E(\tilde{f}(x)) = \frac{(\alpha n)^n}{\Gamma(n)x} \int_0^\infty \frac{e^{wln(\frac{k}{x})}e^{-\alpha n/w}}{w^n} dw.$$

We know $e^{w \ln(\frac{k}{x})} = \sum_{j=0}^{\infty} \frac{w^j (\ln \frac{k}{x})^j}{j!}$

$$E(\tilde{f}(x)) = \frac{(\alpha n)^n}{\Gamma(n)x} \int_0^\infty \sum_{j=0}^\infty \frac{w^j (\ln(\frac{k}{x}))^j}{j! w^n} e^{-\alpha n/w} dw$$
$$= \frac{(\alpha n)^n}{\Gamma(n)x} \sum_{j=0}^\infty \frac{(\ln(\frac{k}{x}))^j}{j!} \int_0^\infty \frac{e^{-\alpha n/w}}{w^{n-j}} dw.$$

Put $\frac{1}{w} = z$ then $\frac{-1}{w^2}dw = dz \Rightarrow dw = \frac{-dz}{z^2}$.

$$E(\tilde{f}(x)) = \frac{(\alpha n)^n}{\Gamma(n)x} \sum_{j=0}^{\infty} \frac{(\ln(\frac{k}{x}))^j}{j!} \int_0^{\infty} z^{n-j-2} e^{-\alpha n z} dz$$

$$= \frac{(\alpha n)^n}{\Gamma(n)x} \sum_{j=0}^{\infty} \frac{(\ln(\frac{k}{x}))^j}{j!} \frac{\Gamma(n-j-1)}{(\frac{1}{n\alpha})^{-n+j+1}} \int_0^{\infty} \frac{z^{n-j-2} e^{-\alpha n z}}{\Gamma(n-j-1)(\frac{1}{n\alpha})^{n-j-1}} dz$$

$$= \frac{1}{\Gamma(n)x} \sum_{j=0}^{\infty} \frac{(n\alpha)^{j+1}}{j!} \Gamma(n-j-1) \left(\ln\left(\frac{k}{x}\right)\right)^j$$

$$\Rightarrow E(\tilde{f}(x)) = \frac{1}{\Gamma(n)x} \sum_{j=0}^{n-2} \frac{(n\alpha)^{j+1}}{j!} \Gamma(n-j-1) \left(\ln\left(\frac{k}{x}\right)\right)^j. \tag{5}$$

(B)

$$E(\tilde{F}(x)) = \int \tilde{F}(x)g(w)dw = \int_0^\infty \left[1 - (\frac{k}{x})^w\right] \frac{(\alpha n)^n}{\Gamma(n)w^{n+1}} e^{-\alpha n/w} dw$$

$$= \int_0^\infty \frac{(\alpha n)^n}{\Gamma(n)w^{n+1}} e^{-\alpha n/w} dw - \frac{(\alpha n)^n}{\Gamma(n)} \int_0^\infty (\frac{k}{x})^w \frac{1}{w^{n+1}} e^{-\alpha n/w} dw$$

$$= 1 - \frac{(\alpha n)^n}{\Gamma(n)} \int_0^\infty \frac{e^{w \ln(\frac{k}{x})} e^{-\alpha n/w}}{w^{n+1}} dw.$$

We know that $\left(\frac{k}{x}\right)^w = e^{w \ln\left(\frac{k}{x}\right)} = \sum_{j=0}^{\infty} \frac{w^j (\ln\left(\frac{k}{x}\right))^j}{j!}$, then

$$E(\tilde{F}(x)) = 1 - \frac{(\alpha n)^n}{\Gamma(n)} \int_0^\infty \sum_{j=0}^\infty \frac{w^j (\ln(\frac{k}{x}))^j}{j!} w^{-n-1} e^{-\alpha n/w} dw$$

$$= 1 - \frac{(\alpha n)^n}{\Gamma(n)} \sum_{j=0}^\infty \frac{(\ln(\frac{k}{x}))^j}{j!} \int_0^\infty w^{-n-1+j} e^{-\alpha n/w} dw$$

$$= 1 - \frac{(\alpha n)^n}{\Gamma(n)} \sum_{j=0}^\infty \frac{(\ln(\frac{k}{x}))^j}{j!} \int_0^\infty \frac{e^{-\alpha n/w}}{w^{n+1-j}} dw.$$

Put $\frac{1}{w} = z \Rightarrow \frac{-1}{w^2} dw = dz \Rightarrow dw = \frac{-1}{z^2} dz$. Then

$$E(\tilde{F}(x)) = 1 - \frac{(\alpha n)^n}{\Gamma(n)} \sum_{j=0}^{\infty} \frac{(\ln(\frac{k}{x}))^j}{j!} \Gamma(n-j) \left(\frac{1}{\alpha n}\right)^{n-j} \int_0^{\infty} \frac{z^{n-j-1}e^{-\alpha nz}}{\Gamma(n-j)(\frac{1}{\alpha n})^{n-j}} dz$$

$$\Rightarrow E(\tilde{F}(x)) = 1 - \frac{1}{\Gamma(n)} \sum_{j=0}^{n-1} \frac{(\alpha n)^j}{j!} \Gamma(n-j) \left(\ln\left(\frac{k}{x}\right)\right)^j, \quad x \ge k.$$
(6)

Proof of the Theorem 2.:

(A) At first, we should find $E(\tilde{f}(x))^2$. So

$$\begin{split} E(\tilde{f}(x))^2 &= \int_0^\infty (\tilde{f}(x))^2 g(w) dw = \int_0^\infty \frac{w^2 k^{2w}}{x^{2w+2}} \frac{(\alpha n)^n}{\Gamma(n) w^{n+1}} e^{-\alpha n/w} dw \\ &= \frac{(\alpha n)^n}{\Gamma(n) x^2} \int_0^\infty \left(\frac{k}{x}\right)^{2w} \frac{e^{-\alpha n/w}}{w^{n-1}} dw. \end{split}$$

Similarly to the pervious Theorem, we have

$$\begin{split} E(\tilde{f}(x))^2 &= \frac{(\alpha n)^n}{\Gamma(n)x^2} \int_0^\infty \frac{e^{2w \ln\left(\frac{k}{x}\right)}e^{-\alpha n/w}}{w^{n-1}} dw \\ &= \frac{(\alpha n)^n}{\Gamma(n)x^2} \int_0^\infty \sum_{j=0}^\infty \frac{2^j w^j (\ln\left(\frac{k}{x}\right))^j}{j!w^{n-1}} e^{-\alpha n/w} dw \\ &= \frac{(\alpha n)^n}{\Gamma(n)x^2} \sum_{j=0}^\infty \frac{2^j (\ln\left(\frac{k}{x}\right))^j}{j!} \int_0^\infty \frac{e^{-\alpha n/w}}{w^{n-j-1}} dw \\ &\stackrel{\frac{1}{w}=z}{=} \frac{(\alpha n)^n}{\Gamma(n)x^2} \sum_{j=0}^\infty \frac{2^j (\ln\left(\frac{k}{x}\right))^j}{j!} \int_0^\infty z^{n-j-3} e^{-\alpha nz} dz \\ &= \frac{(\alpha n)^n}{\Gamma(n)x^2} \sum_{j=0}^\infty \frac{2^j (\ln\left(\frac{k}{x}\right))^j}{j!} \Gamma(n-j-2) \left(\frac{1}{\alpha n}\right)^{n-j-2} \int_0^\infty \frac{z^{n-j-3} e^{-\alpha nz}}{\Gamma(n-j-2)(\frac{1}{\alpha n})^{n-j-2}} dz \\ \Rightarrow E(\tilde{f}(x))^2 &= \frac{1}{\Gamma(n)x^2} \sum_{j=0}^{n-2} \frac{2^j (\ln\left(\frac{k}{x}\right))^j}{j!} \Gamma(n-j-2)(\alpha n)^{j+2}. \end{split}$$

We know that $V(\tilde{f}(x)) = E(\tilde{f}(x))^2 - E^2(\tilde{f}(x))$. Then

$$V(\tilde{f}(x)) = \frac{1}{\Gamma(n)x^2} \sum_{j=0}^{n-2} \frac{2^j (\ln\left(\frac{k}{x}\right))^j}{j!} \Gamma(n-j-2) (\alpha n)^{j+2} - \left[\frac{1}{\Gamma(n)x} \sum_{j=0}^{n-1} \frac{(\alpha n)^{j+1}}{j!} \Gamma(n-j-1) (\ln\left(\frac{k}{x}\right))^j \right]^2.$$

Therefore

$$MSE(\tilde{f}(x)) = V(\tilde{f}(x)) + (E(\tilde{f}(x)) - f(x))^{2}$$

$$= V(\tilde{f}(x)) + E^{2}(\tilde{f}(x)) - 2E(\tilde{f}(x))f(x) + f^{2}(x)$$

$$= E(\tilde{f}(x))^{2} - E^{2}(\tilde{f}(x)) + E^{2}(\tilde{f}(x)) - 2E(\tilde{f}(x))f(x) + f^{2}(x)$$

$$= E(\tilde{f}(x))^{2} - 2f(x)E(\tilde{f}(x)) + f^{2}(x)$$

$$\Rightarrow MSE(\tilde{f}(x)) = \frac{1}{\Gamma(n)x^{2}} \sum_{j=0}^{n-2} \frac{2^{j}(\ln(\frac{k}{x}))^{j}}{j!} \Gamma(n-j-2)(\alpha n)^{j+2}$$

$$- 2\frac{\alpha k^{\alpha}}{x^{\alpha+1}} \frac{1}{\Gamma(n)x} \sum_{j=0}^{n-1} \frac{(n\alpha)^{j+1}}{j!} \Gamma(n-j-1)(\ln(\frac{k}{x}))^{j} + (\frac{\alpha k^{\alpha}}{x^{\alpha+1}})^{2}.$$
 (7)

(B) Seme as the case (A)

$$E(\tilde{F}(x))^2 = \int_0^\infty (\tilde{F}(x))^2 g(w) dw = \int_0^\infty \left[1 - \left(\frac{k^w}{x^w} \right) \right]^2 \frac{(\alpha n)^n}{\Gamma(n) w^{n+1}} e^{-\alpha n/w} dw$$

$$= \int_{0}^{\infty} \frac{(\alpha n)^{n}}{\Gamma(n)w^{n+1}} e^{-\alpha n/w} dw - 2\frac{(\alpha n)^{n}}{\Gamma(n)} \int_{0}^{\infty} \frac{(\frac{k}{x})^{w}e^{-\alpha n/w}}{w^{n+1}} dw + \frac{(\alpha n)^{n}}{\Gamma(n)} \int_{0}^{\infty} \frac{(\frac{k}{x})^{2w}e^{-\alpha n/w}}{w^{n+1}} dw$$

$$= 1 - 2\frac{(\alpha n)^{n}}{\Gamma(n)} \int_{0}^{\infty} \frac{e^{w \ln(\frac{k}{x})}e^{-\alpha n/w}}{w^{n+1}} dw + \frac{(\alpha n)^{n}}{\Gamma(n)} \int_{0}^{\infty} \frac{e^{2w \ln(\frac{k}{x})e^{-\alpha n/w}}}{w^{n+1}} dw$$

$$\text{Let } e^{w \ln(\frac{k}{x})} = \sum_{j=0}^{\infty} \frac{w^{j}(\ln(\frac{k}{x}))^{j}}{j!}, \quad e^{2w \ln(\frac{k}{x})} = \sum_{j=0}^{\infty} \frac{2^{j}w^{j}(\ln(\frac{k}{x}))^{j}}{j!}, \text{ then }$$

$$E(\tilde{F}(x))^{2} = 1 - 2\frac{(\alpha n)^{n}}{\Gamma(n)} \int_{0}^{\infty} \sum_{j=0}^{\infty} \frac{w^{j}(\ln(\frac{k}{x}))^{j}}{j!} \frac{e^{-\alpha n/w}}{w^{n+1}} dw$$

$$+ \frac{(\alpha n)^{n}}{\Gamma(n)} \int_{0}^{\infty} \sum_{j=0}^{\infty} \frac{2^{j}w^{j}(\ln(\frac{k}{x}))^{j}}{j!} \frac{e^{-\alpha n/w}}{w^{n+1}} dw$$

$$= 1 - 2\frac{(\alpha n)^{n}}{\Gamma(n)} \sum_{j=0}^{\infty} \frac{(\ln(\frac{k}{x}))^{j}}{j!} \int_{0}^{\infty} \frac{e^{-\alpha n/w}}{w^{n+1-j}} dw$$

$$+ \frac{(\alpha n)^{n}}{\Gamma(n)} \sum_{j=0}^{\infty} \frac{2^{j}(\ln(\frac{k}{x}))^{j}}{j!} \int_{0}^{\infty} \frac{e^{-\alpha n/w}}{w^{n+1-j}} dw$$

$$\frac{1}{w} = 1 - 2\frac{(\alpha n)^{n}}{\Gamma(n)} \sum_{j=0}^{\infty} \frac{(\ln(\frac{k}{x}))^{j}}{j!} \Gamma(n-j) \left(\frac{1}{\alpha n}\right)^{n-j} \int_{0}^{\infty} \frac{z^{n-1-j}e^{-\alpha nz}}{\Gamma(n-j)(\frac{1}{\alpha n})^{n-j}} dz$$

$$+ \frac{(\alpha n)^{n}}{\Gamma(n)} \sum_{j=0}^{\infty} \frac{2^{j}(\ln(\frac{k}{x}))^{j}}{j!} \Gamma(n-j) \left(\frac{1}{\alpha n}\right)^{n-j} \int_{0}^{\infty} \frac{z^{n-1-j}e^{-\alpha nz}}{\Gamma(n-j)(\frac{1}{\alpha n})^{n-j}} dz$$

$$\Rightarrow E(\tilde{F}(x))^{2} = 1 - \frac{2}{\Gamma(n)} \sum_{j=0}^{n-1} \frac{\Gamma(n-j)(\alpha n)^{j}}{j!} (\ln(\frac{k}{x}))^{j}, \quad x \ge k.$$

We have $V(\tilde{F}(x)) = E(\tilde{F}(x))^2 - E^2(\tilde{F}(x))$, so

$$V(\tilde{F}(x)) = 1 - \frac{2}{\Gamma(n)} \sum_{j=0}^{n} \frac{\Gamma(n-j)(\alpha n)^{j}}{j!} (\ln\left(\frac{k}{x}\right))^{j} + \frac{1}{\Gamma(n)} \sum_{j=0}^{n} \frac{\Gamma(n-j)2^{j}(\alpha n)^{j}}{j!} (\ln\left(\frac{k}{x}\right))^{j}$$
$$- \left[1 - \frac{1}{\Gamma(n)} \sum_{j=0}^{n} \frac{\Gamma(n-j)(\alpha n)^{j}}{j!} (\ln\left(\frac{k}{x}\right))^{j}\right]^{2}.$$

Further $MSE(\tilde{F}(x)) = V(\tilde{F}(x)) + (E(\tilde{F}(x)) - F(x))^2 = E(\tilde{F}(x))^2 - 2F(x)E(\tilde{F}(x)) + F^2(x)$, then

$$MSE(\tilde{F}(x)) = 1 - \frac{2}{\Gamma(n)} \sum_{j=0}^{n} \frac{\Gamma(n-j)(\alpha n)^{j}}{j!} (\ln\left(\frac{k}{x}\right))^{j} + \frac{1}{\Gamma(n)} \sum_{j=0}^{n} \frac{\Gamma(n-j)2^{j}(\alpha n)^{j}}{j!} (\ln\left(\frac{k}{x}\right))^{j}$$
$$- 2\left[1 - \left(\frac{k}{x}\right)^{\alpha}\right] \left[1 - \frac{1}{\Gamma(n)} \sum_{j=0}^{n} \frac{\Gamma(n-j)(\alpha n)^{j}}{j!} (\ln\left(\frac{k}{x}\right))^{j}\right] + \left[1 - \left(\frac{k}{x}\right)^{\alpha}\right]^{2}.$$

Therefore

$$MSE(\tilde{F}(x)) = 2 + \frac{1}{\Gamma(n)} \sum_{j=0}^{n} \frac{\Gamma(n-j)2^{j}(\alpha n)^{j}}{j!} (\ln\left(\frac{k}{x}\right))^{j}$$

$$-2\left(\frac{k}{x}\right)^{\alpha} \frac{1}{\Gamma(n)} \sum_{j=0}^{n} \frac{\Gamma(n-j)(\alpha n)^{j}}{j!} \left(\ln\left(\frac{k}{x}\right)\right)^{j} + \left(\frac{k}{x}\right)^{2\alpha}.$$
 (8)

From Asrabadi (1990), we have

$$\hat{\alpha} = \frac{n-1}{\ln(t) - n \ln(k)}, \quad t \ge k^n, \tag{9}$$

the UMVUE of f(x) and F(x) is

$$\hat{f}(x) = \frac{(n-1)[\ln(t) - \ln(x) - (n-1)\ln(k)]^{n-2}}{x[\ln(t) - n\ln(k)]^{n-1}}, \quad k \le x < tk^{1-n},$$
(10)

$$\hat{F}(x) = \begin{cases} 0 & x < k, \\ 1 - \frac{[\ln(t) - \ln(x) - (n-1)\ln(k)]^{n-1}}{[\ln(t) - n\ln(k)]^{n-1}} & k \le x \le tk^{1-n}, \\ 1 & x \ge tk^{1-n}, \end{cases}$$
(11)

respectively. Also,

$$f(x) = \frac{\alpha k^{\alpha}}{x^{\alpha+1}}, \quad x \ge k > 0, \ \alpha > 0,$$

$$F(x) = 1 - \left(\frac{k}{x}\right)^{\alpha}, \quad k \le x.$$

Proof of the Theorem 3.:

(A) It is obvious that E(f(x)) = f(x). So, we should find

$$E(\hat{f}(x))^2 = \int \hat{f}^2(x)h^*(x)dt,$$

where,

$$h^*(x) = \frac{\alpha^n k^{n\alpha}}{(n-1)!} t^{-\alpha - 1} [\ln(t) - n \ln(k)]^{n-1}, \quad t \ge k^n.$$

Therefore

$$\begin{split} E(\hat{f}(x))^2 &= \int_{xk^{n-1}}^{\infty} \frac{(n-1)^2 [\ln(t) - \ln(x) - (n-1)\ln(k)]^{2n-4}}{x^2 [\ln(t) - n\ln(k)]^{2n-2}} \frac{\alpha^n k^{n\alpha}}{(n-1)!} t^{-\alpha-1} [\ln(t) - n\ln(k)]^{n-1} dt \\ &= \frac{(n-1)^2 \alpha^n k^{\alpha n}}{x^2 (n-1)!} \int_{xk^{n-1}}^{\infty} \frac{[\ln(t) - \ln(x) - (n-1)\ln(k)]^{2n-4}}{[\ln(t) - n\ln(k)]^{n-1}} t^{-\alpha-1} dt. \end{split}$$

Let $z = \ln(t) - n \ln(k) \Rightarrow dz = \frac{1}{t}dt$, then

$$E(\hat{f}(x))^2 = \frac{(n-1)\alpha^n}{x^2(n-2)!} \int_{\ln(\frac{x}{k})}^{\infty} \frac{[z - \ln(x) + \ln(k)]^{2n-4} e^{-\alpha z}}{z^{n-1}} dz.$$

We know that

$$[z - \ln(x) + \ln(k)]^{2n-4} = \sum_{j=0}^{2n-4} C(2n-4, j) \left(-\ln\left(\frac{x}{k}\right)\right)^j z^{2n-4-j},$$

where $C(n,k) = \frac{n!}{k!(n-k)!}$. So

$$E(\hat{f}(x))^2 = \frac{(n-1)\alpha^n}{x^2(n-2)!} \sum_{j=0}^{2n-4} C(2n-4,j) \left(-\ln\left(\frac{x}{k}\right)\right)^j \int_{\ln\left(\frac{x}{k}\right)}^{\infty} z^{n-3-j} e^{-\alpha z} dz.$$

The above integral is the incomplete Gamma function, therefore

$$\begin{split} E(\hat{f}(x))^2 &= \frac{(n-1)\alpha^n}{x^2(n-2)!} \sum_{j=0}^{2n-4} C(2n-4,j) \left(-\ln\left(\frac{x}{k}\right)\right)^j \\ &\times \frac{\Gamma(n-2-j)}{\alpha^{n-2-j}} \sum_{i=0}^{n-3-j} \frac{\exp\left(-\alpha\ln\left(\frac{x}{k}\right)\right) \left(\alpha\ln\left(\frac{x}{k}\right)\right)^i}{i!} \\ &= \frac{(n-1)\alpha^2 k^\alpha}{x^{\alpha+2}(n-2)!} \sum_{j=0}^{2n-4} C(2n-4,j)\alpha^j \left[-\ln\left(\frac{x}{k}\right)\right]^j \Gamma(n-2-j) \sum_{i=0}^{n-3-j} \frac{\alpha^i \left(\ln\left(\frac{x}{k}\right)\right)^i}{i!}. \end{split}$$

We know that the Gamma function is defined on the positive value. So

$$E(\hat{f}(x))^{2} = \frac{(n-1)\alpha^{2}k^{\alpha}}{x^{\alpha+2}(n-2)!} \sum_{j=0}^{n-3} C(2n-4,j)\alpha^{j} \left[-\ln\left(\frac{x}{k}\right)\right]^{j} \Gamma(n-2-j) \sum_{i=0}^{n-3-j} \frac{\alpha^{i} \left(\ln\left(\frac{x}{k}\right)\right)^{i}}{i!}.$$

Finally

$$MSE(\hat{f}(x)) = V(\hat{f}(x)) = \frac{(n-1)\alpha^2 k^{\alpha}}{x^{\alpha+2}\Gamma(n-1)} \sum_{j=0}^{n-3} C(2n-4,j)\alpha^j \Gamma(n-j-2) \left(-\ln\left(\frac{x}{k}\right)\right)^j \times \sum_{i=0}^{n-3-j} \frac{\alpha^i \left(\ln\left(\frac{x}{k}\right)\right)^i}{i!} - \left(\frac{\alpha k^{\alpha}}{x^{\alpha+1}}\right)^2.$$

$$(12)$$

(B)

$$\begin{split} E(\hat{F}(x))^2 &= \int \hat{F}^2(x)h^*(t)dt \\ &= \int_{xk^{n-1}}^{\infty} \left[1 - \frac{[\ln(t) - \ln(x) - (n-1)\ln(k)]^{n-1}}{[\ln(t) - n\ln(k)]^{n-1}}\right]^2 h^*(t)dt + \int_{k^n}^{xk^{n-1}} 1^2 \times h^*(t)dt \\ &= \int_{xk^{n-1}}^{\infty} h^*(t)dt - 2\int_{xk^{n-1}}^{\infty} \frac{[\ln(t) - \ln(x) - (n-1)\ln(k)]^{n-1}}{[\ln(t) - n\ln(k)]^{n-1}} h^*(t)dt \\ &+ \int_{xk^{n-1}}^{\infty} \frac{[\ln(t) - \ln(x) - (n-1)\ln(k)]^{2n-2}}{[\ln(t) - n\ln(k)]^{2n-2}} h^*(t)dt + \int_{k^n}^{xk^{n-1}} h^*(t)dt \\ &= \int_{xk^{n-1}}^{\infty} h^*(t)dt + \int_{k^n}^{xk^{n-1}} h^*(t)dt - 2\int_{xk^{n-1}}^{\infty} \frac{[\ln(t) - \ln(x) - (n-1)\ln(k)]^{n-1}}{[\ln(t) - n\ln(k)]^{n-1}} \\ &\times \frac{\alpha^n k^{\alpha n}}{(n-1)!} t^{-\alpha-1} [\ln(t) - n\ln(k)]^{n-1} dt \\ &+ \int_{xk^{n-1}}^{\infty} \frac{[\ln(t) - \ln(x) - (n-1)\ln(k)]^{2n-2}}{[\ln(t) - n\ln(k)]^{2n-2}} \frac{\alpha^n k^{\alpha n}}{(n-1)!} t^{-\alpha-1} [\ln(t) - n\ln(k)]^{n-1} dt \\ &= \int_{k^n}^{\infty} h^*(t)dt - 2\frac{\alpha^n k^{\alpha n}}{(n-1)!} \int_{xk^{n-1}}^{\infty} [\ln(t) - \ln(x) - (n-1)\ln(k)]^{2n-2}} t^{-\alpha-1} dt \\ &+ \frac{\alpha^n k^{\alpha n}}{(n-1)!} \int_{xk^{n-1}}^{\infty} \frac{[\ln(t) - \ln(x) - (n-1)\ln(k)]^{2n-2}}{[\ln(t) - n\ln(k)]^{n-1}} t^{-\alpha-1} dt. \end{split}$$

We know that $\int_{k^n}^{\infty} h^*(t)dt = 1$. For second part let $z = \ln(t) - \ln(x) - (n-1)\ln(k)$ and to solve the third integral put $z = \ln(t) - n\ln(k)$. Then

$$\begin{split} E(\hat{F}(x))^2 &= 1 - 2 \frac{\alpha^n k^{\alpha n}}{(n-1)!} x^{-\alpha} k^{-\alpha(n-1)} \int_0^\infty z^{n-1} e^{-\alpha z} dz \\ &+ \frac{\alpha^n k^{\alpha n}}{(n-1)!} k^{-n\alpha} \int_{\ln\left(\frac{x}{k}\right)}^\infty \frac{[z - \ln(x) + \ln(k)]^{2n-2}}{z^{n-1}} e^{-\alpha z} dz \\ &= 1 - 2 \frac{\alpha^n k^{\alpha}}{x^{\alpha}(n-1)!} \int_0^\infty z^{n-1} e^{-\alpha z} dz + \frac{\alpha^n}{(n-1)!} \int_{\ln\left(\frac{x}{k}\right)}^\infty \frac{[z - \ln(x) + \ln(k)]^{2n-2} e^{-\alpha z}}{z^{n-1}} dz. \end{split}$$

 $\int_0^\infty z^{n-1}e^{-\alpha z}dz=\frac{\Gamma(n)}{\alpha^n}$ and for the last integral, we know $[z-\ln(x)+\ln(k)]^{2n-2}=\sum_{j=0}^{2n-2}C(2n-2,j)z^{2n-2-j}\left[-\ln\left(\frac{x}{k}\right)\right]^j$. Therefore

$$\begin{split} E(\hat{F}(x))^2 &= 1 - 2\frac{k^{\alpha}}{x^{\alpha}} + \frac{\alpha^n}{(n-1)!} \sum_{j=0}^{2n-2} C(2n-2,j) \left[-\ln\left(\frac{x}{k}\right) \right]^j \int_{\ln\left(\frac{x}{k}\right)}^{\infty} z^{n-1-j} e^{-\alpha z} z^{n-1} dz \\ &= 1 - 2\frac{k^{\alpha}}{x^{\alpha}} + \frac{\alpha^n}{(n-1)!} \sum_{j=0}^{2n-2} C(2n-2,j) \left[-\ln\left(\frac{x}{k}\right) \right]^j \frac{\Gamma(n-j)}{\alpha^{n-j}} \int_{\ln\left(\frac{x}{k}\right)}^{\infty} \frac{\alpha^{n-j} z^{n-1-j} e^{-\alpha z}}{\Gamma(n-j)} dz. \end{split}$$

The last integral is the incomplete Gamma function, then

$$E(\hat{F}(x))^{2} = 1 - 2\frac{k^{\alpha}}{x^{\alpha}} + \frac{\alpha^{n}}{(n-1)!} \sum_{j=0}^{2n-2} C(2n-2,j) \left[-\ln\left(\frac{x}{k}\right)\right]^{j}$$

$$\times \frac{\Gamma(n-j)}{\alpha^{n-j}} \sum_{i=0}^{n-1-j} \frac{e^{-\alpha \ln\left(\frac{x}{k}\right)} \left[\alpha \ln\left(\frac{x}{k}\right)\right]^{i}}{i!}.$$

The Gamma function is defined over positive value, So

$$E(\hat{F}(x))^{2} = 1 - 2\frac{k^{\alpha}}{x^{\alpha}} + \frac{k^{\alpha}}{x^{\alpha}(n-1)!} \sum_{j=0}^{n-1} C(2n-2,j) \left[-\ln\left(\frac{x}{k}\right) \right]^{j}$$
$$\times \alpha^{j} \Gamma(n-j) \sum_{i=0}^{n-1-j} \frac{\left[\alpha \ln\left(\frac{x}{k}\right)\right]^{i}}{i!}.$$

Then

$$\begin{split} MSE(\hat{F}(x)) &= V(\hat{F}(x)) = E(\hat{F}(x))^2 - E^2(\hat{F}(x)) = E(\hat{F}(x))^2 - F^2(x) \\ &= 1 - 2\frac{k^\alpha}{x^\alpha} + \frac{k^\alpha}{x^\alpha(n-1)!} \sum_{j=0}^{n-1} C(2n-2,j) \left[-\ln\left(\frac{x}{k}\right) \right]^j \\ &\times \alpha^j \Gamma(n-j) \sum_{i=0}^{n-1-j} \frac{\left[\alpha \ln\left(\frac{x}{k}\right)\right]^i}{i!} - \left[1 - \left(\frac{k}{x}\right)^\alpha\right]^2. \end{split}$$

Finally

$$MSE(\hat{F}(x)) = \frac{k^{\alpha}}{\Gamma(n)x^{\alpha}} \sum_{j=0}^{n-1} C(2n-2,j)\alpha^{j} \Gamma(n-j) \left[-\ln\left(\frac{x}{k}\right)\right]^{j}$$

$$\times \sum_{i=0}^{n-1-j} \frac{\alpha^{i} \left[\ln\left(\frac{x}{k}\right)\right]^{i}}{i!} - \left(\frac{k}{x}\right)^{2\alpha}.$$
(13)

2.1 The rth estimate of $\tilde{f}(x)$ and $\tilde{F}(x)$

To find the rth estimate of $\tilde{f}(x)$, we have

$$E(\tilde{f}(x))^{r} = \int_{0}^{\infty} (\tilde{f}(x))^{r} g(w) dw = \int_{0}^{\infty} \frac{w^{r} k^{rw}}{x^{r(w+1)}} \frac{(\alpha n)^{n}}{\Gamma(n) w^{n+1}} e^{-\alpha n/w} dw$$

$$= \frac{(\alpha n)^{n}}{\Gamma(n) x^{r}} \int_{0}^{\infty} \left(\frac{k}{x}\right)^{rw} \frac{e^{-\alpha n/w}}{w^{n-r+1}} dw$$

$$= \frac{(\alpha n)^{n}}{\Gamma(n) x^{r}} \int_{0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{rw \ln\left(\frac{k}{x}\right)} e^{-\alpha n/w}}{w^{n-r+1}} dw$$

$$= \frac{(\alpha n)^{n}}{\Gamma(n) x^{r}} \int_{0}^{\infty} \sum_{j=0}^{\infty} \frac{r^{j} w^{j} (\ln\left(\frac{k}{x}\right))^{j}}{j! w^{n-r+1}} e^{-\alpha n/w} dw$$

$$= \frac{(\alpha n)^{n}}{\Gamma(n) x^{r}} \sum_{j=0}^{\infty} \frac{r^{j} (\ln\left(\frac{k}{x}\right))^{j}}{j!} \int_{0}^{\infty} \frac{e^{-\alpha n/w}}{w^{n-r-j+1}} dw$$

$$\stackrel{\frac{1}{w}=z}{=} \frac{(\alpha n)^{n}}{\Gamma(n) x^{r}} \sum_{j=0}^{\infty} \frac{r^{j} (\ln\left(\frac{k}{x}\right))^{j}}{j!} \int_{0}^{\infty} z^{n-r-j-1} e^{-\alpha n z} dz$$

$$= \frac{(\alpha n)^{n}}{\Gamma(n) x^{r}} \sum_{j=0}^{\infty} \frac{r^{j} (\ln\left(\frac{k}{x}\right))^{j}}{j!} \Gamma(n-r-j) \left(\frac{1}{\alpha n}\right)^{n-r-j}$$

$$\Rightarrow E(\tilde{f}(x))^{r} = \frac{1}{\Gamma(n) x^{r}} \sum_{j=0}^{n-r-1} \frac{r^{j} (\ln\left(\frac{k}{x}\right))^{j}}{j!} \Gamma(n-r-j) (\alpha n)^{j+r}. \tag{14}$$

Also, the rth estimate of $\tilde{F}(x)$ can be found by calculating the following integral

$$\begin{split} E(\tilde{F}(x))^r &= \int_0^\infty (\tilde{F}(x))^r g(w) dw = \int_0^\infty \left[1 - \left(\frac{k}{x} \right)^w \right]^r \frac{(\alpha n)^n}{\Gamma(n) w^{n+1}} e^{-\alpha n/w} dw \\ &= \int_0^\infty \sum_{j=0}^r C(r,j) \left(- \left(\frac{k}{x} \right)^w \right)^j \frac{(\alpha n)^n}{\Gamma(n) w^{n+1}} e^{-\alpha n/w} dw \\ &= \frac{(\alpha n)^n}{\Gamma(n)} \sum_{j=0}^r C(r,j) (-1)^j \int_0^\infty \frac{(\frac{k}{x})^{jw} e^{-\alpha n/w}}{w^{n+1}} dw \\ &= \frac{(\alpha n)^n}{\Gamma(n)} \sum_{j=0}^r C(r,j) (-1)^j \int_0^\infty \frac{e^{jw \ln\left(\frac{k}{x}\right)} e^{-\alpha n/w}}{w^{n+1}} dw \\ &= \frac{(\alpha n)^n}{\Gamma(n)} \sum_{j=0}^r C(r,j) (-1)^j \sum_{i=0}^\infty \frac{\left(j \ln\left(\frac{k}{x}\right)\right)^i}{i!} \int_0^\infty \frac{e^{-\alpha n/w}}{w^{n+1-i}} dw \\ &\stackrel{\frac{1}{w}=z}{=} \frac{(\alpha n)^n}{\Gamma(n)} \sum_{j=0}^r C(r,j) (-1)^j \sum_{i=0}^\infty \frac{\left(j \ln\left(\frac{k}{x}\right)\right)^i}{i!} \int_0^\infty z^{n-i-1} e^{-\alpha nz} dz \\ &= \frac{(\alpha n)^n}{\Gamma(n)} \sum_{j=0}^r C(r,j) (-1)^j \sum_{i=0}^\infty \frac{\left(j \ln\left(\frac{k}{x}\right)\right)^i}{i!} \Gamma(n-i) (\frac{1}{\alpha n})^{n-i}. \end{split}$$

Then

$$E(\tilde{F}(x))^{r} = \frac{1}{\Gamma(n)} \sum_{i=0}^{r} C(r,j) (-1)^{j} \sum_{i=0}^{n-1} \frac{\left(j \ln\left(\frac{k}{x}\right)\right)^{i}}{i!} \Gamma(n-i) (\alpha n)^{i}.$$
 (15)

2.2 The rth estimate of $\hat{f}(x)$ and $\hat{F}(x)$

The rth estimate of $\hat{f}(x)$ is easily obtained as follows.

$$\begin{split} E(\hat{f}(x))^r &= \int (\hat{f}(x))^r \frac{\alpha^n k^{n\alpha}}{(n-1)!} t^{-\alpha-1} [\ln(t) - n \ln(k)]^{n-1} dt \\ &= \int_{xk^{n-1}}^{\infty} \frac{(n-1)^r [\ln(t) - \ln(x) - (n-1) \ln(k)]^{r(n-2)}}{x^r [\ln(t) - n \ln(k)]^{r(n-1)}} \frac{\alpha^n k^{n\alpha}}{(n-1)!} t^{-\alpha-1} [\ln(t) - n \ln(k)]^{n-1} dt \\ &= \frac{(n-1)^r \alpha^n k^{\alpha n}}{x^r (n-1)!} \int_{xk^{n-1}}^{\infty} \frac{[\ln(t) - \ln(x) - (n-1) \ln(k)]^{r(n-2)}}{[\ln(t) - n \ln(k)]^{(r-1)(n-1)}} t^{-\alpha-1} dt. \end{split}$$

Let $z = \ln(t) - n \ln(k) \Rightarrow dz = \frac{1}{t}dt$, then

$$\begin{split} E(\hat{f}(x))^r &= \frac{(n-1)^r \alpha^n k^{\alpha n}}{x^r (n-1)!} \int_{\ln\left(\frac{x}{k}\right)}^{\infty} \frac{[z-\ln(x)+\ln(k)]^{r(n-2)} k^{-\alpha n} e^{-\alpha z}}{z^{(r-1)(n-1)}} dz \\ &= \frac{(n-1)^r \alpha^n}{x^r (n-1)!} \int_{\ln\left(\frac{x}{k}\right)}^{\infty} \frac{\sum_{j=0}^{r(n-2)} C(r(n-2),j) \left(-\ln\left(\frac{x}{k}\right)\right)^j z^{r(n-2)-j} e^{-\alpha z}}{z^{(r-1)(n-1)}} dz \\ &= \frac{(n-1)^r \alpha^n}{x^r (n-1)!} \sum_{j=0}^{r(n-2)} C(r(n-2),j) \left(-\ln\left(\frac{x}{k}\right)\right)^j \int_{\ln\left(\frac{x}{k}\right)}^{\infty} z^{n-r-j-1} e^{-\alpha z} dz \\ &= \frac{(n-1)^r \alpha^n}{x^r (n-1)!} \sum_{j=0}^{r(n-2)} C(r(n-2),j) \left(-\ln\left(\frac{x}{k}\right)\right)^j \\ &\times \frac{\Gamma(n-r-j)}{\alpha^{n-r-j}} \sum_{j=0}^{n-r-j-1} \frac{\exp\left(-\alpha \ln\left(\frac{x}{k}\right)\right) \left(\alpha \ln\left(\frac{x}{k}\right)\right)^i}{i!} \end{split}$$

Therefore

$$E(\hat{f}(x))^{r} = \frac{(n-1)^{r} \alpha^{r} k^{\alpha}}{x^{\alpha+r} (n-1)!} \sum_{i=0}^{n-r-1} C(r(n-2), j) \alpha^{j} \left[-\ln\left(\frac{x}{k}\right) \right]^{j} \Gamma(n-r-j) \sum_{i=0}^{n-r-j-1} \frac{\alpha^{i} \left(\ln\left(\frac{x}{k}\right)\right)^{i}}{i!}.$$
 (16)

Also, the rth estimate of $\hat{F}(x)$ is similarly obtained as follows.

$$\begin{split} E(\hat{F}(x))^r &= \int (\hat{F}(x))^r h^*(t) dt \\ &= \int_{xk^{n-1}}^{\infty} \left[1 - \frac{[\ln(t) - \ln(x) - (n-1)\ln(k)]^{n-1}}{[\ln(t) - n\ln(k)]^{n-1}} \right]^r h^*(t) dt + \int_{k^n}^{xk^{n-1}} 1^r \times h^*(t) dt \\ &= \int_{xk^{n-1}}^{\infty} \sum_{j=0}^{r} C(r,j) (-1)^j \frac{[\ln(t) - \ln(x) - (n-1)\ln(k)]^{j(n-1)}}{[\ln(t) - n\ln(k)]^{j(n-1)}} \\ &\times \frac{\alpha^n k^{n\alpha}}{(n-1)!} t^{-\alpha-1} [\ln(t) - n\ln(k)]^{n-1} dt + \int_{k^n}^{xk^{n-1}} \frac{\alpha^n k^{n\alpha}}{(n-1)!} t^{-\alpha-1} [\ln(t) - n\ln(k)]^{n-1} dt \\ &= \frac{\alpha^n k^{n\alpha}}{(n-1)!} \sum_{j=0}^{r} C(r,j) (-1)^j \int_{xk^{n-1}}^{\infty} \frac{[\ln(t) - \ln(x) - (n-1)\ln(k)]^{j(n-1)}}{[\ln(t) - n\ln(k)]^{(j-1)(n-1)}} t^{-\alpha-1} dt \\ &+ \frac{\alpha^n k^{n\alpha}}{(n-1)!} \int_{k^n}^{xk^{n-1}} t^{-\alpha-1} [\ln(t) - n\ln(k)]^{n-1} dt. \end{split}$$

Let
$$z = \ln(t) - n \ln(k) \Rightarrow dz = \frac{1}{t}dt$$
, then

$$\begin{split} E(\hat{F}(x))^r &= \frac{\alpha^n k^{n\alpha}}{(n-1)!} \sum_{j=0}^r C(r,j) (-1)^j \int_{\ln(\frac{x}{k})}^\infty \frac{[z-\ln(x)+\ln(k)]^{j(n-1)}}{z^{(j-1)(n-1)}} k^{-\alpha n} e^{-\alpha z} dz \\ &+ \frac{\alpha^n k^{n\alpha}}{(n-1)!} \int_0^{\ln(\frac{x}{k})} z^{n-1} k^{-\alpha n} e^{-\alpha z} dz \\ &= \frac{\alpha^n}{(n-1)!} \sum_{j=0}^r C(r,j) (-1)^j \int_{\ln(\frac{x}{k})}^\infty \frac{\sum_{i=0}^{j(n-1)} C(j(n-1),i) (-\ln(\frac{x}{k}))^i z^{j(n-1)-i} e^{-\alpha z}}{z^{(j-1)(n-1)}} dz \\ &+ \frac{\alpha^n}{(n-1)!} \frac{\Gamma(n)}{\alpha^n} \left[1 - \left(\frac{x}{k} \right)^{-\alpha} \sum_{i=0}^{n-1} \frac{(\alpha \ln(\frac{x}{k}))^i}{i!} \right] \\ &= \frac{\alpha^n}{(n-1)!} \sum_{j=0}^r C(r,j) (-1)^j \sum_{i=0}^{j(n-1)} C(j(n-1),i) \left(-\ln\left(\frac{x}{k}\right) \right)^i \int_{\ln(\frac{x}{k})}^\infty z^{n-i-1} e^{-\alpha z} dz \\ &+ \left[1 - \left(\frac{x}{k} \right)^{-\alpha} \sum_{i=0}^{n-1} \frac{(\alpha \ln(\frac{x}{k}))^i}{i!} \right] \\ &= \frac{\alpha^n}{(n-1)!} \sum_{j=0}^r C(r,j) (-1)^j \sum_{i=0}^{j(n-1)} C(j(n-1),i) \left(-\ln\left(\frac{x}{k}\right) \right)^i \frac{\Gamma(n-i)}{\alpha^{n-i}} \\ &\times \sum_{l=0}^{n-i-1} \frac{\exp(-\alpha \ln(\frac{x}{k}))(\alpha \ln(\frac{x}{k}))^l}{l!} + \left(\frac{x}{k} \right)^{-\alpha} \left[\left(\frac{x}{k} \right)^{\alpha} - \sum_{i=0}^{n-1} \frac{(\alpha \ln(\frac{x}{k}))^i}{i!} \right]. \end{split}$$

Then

$$E(\hat{F}(x))^{r} = \frac{k^{\alpha}}{(n-1)!x^{\alpha}} \sum_{j=0}^{r} C(r,j)(-1)^{j} \sum_{i=0}^{j(n-1)} C(j(n-1),i) \left(-\ln\left(\frac{x}{k}\right)\right)^{i} \alpha^{i} \Gamma(n-i)$$

$$\times \sum_{l=0}^{n-i-1} \frac{(\alpha \ln(\frac{x}{k}))^{l}}{l!} + \left(\frac{k}{x}\right)^{\alpha} \left[\left(\frac{x}{k}\right)^{\alpha} - \sum_{i=0}^{n-1} \frac{(\alpha \ln(\frac{x}{k}))^{i}}{i!}\right]. \tag{17}$$

3 R code

The R code to compare the bias and MSE of the estimators is as follows.

```
sim=function(t,n,k,alpha,r)
{
    sfh<-0
    sFh<-0
    sft<-0
    sFt<-0
    for(l in 1:t){
        x<-array(, c(1,n))
        for (i in 1:n) {
        u<-runif(1,0,1)
        x[i]<-k*(1-u)^(-1/alpha)}
        alphah<-n/sum(log(x)-log(k))
    fx<-alpha*k^alpha/x[1]^(alpha+1)
    intB<-function(z){ z^(-n)*exp(-alpha*n/z)*(k/x[1])^z}</pre>
```

```
B<-integrate(intB,lower=0,upper=Inf)$value
Eftildx<-(alpha*n)^n/factorial(n-1)/x[1]*B
   intB1 < -function(z) \{ z^{(-n+1)} * exp(-alpha*n/z) * (k/x[1])^{(2*z)} \}
   B1<-integrate(intB1,lower=0,upper=Inf)$value
   Eftildxs2<-(alpha*n)^n/factorial(n-1)/x[1]^2*B1</pre>
MSEftildx<-Eftildxs2-2*fx*Eftildx+fx^2
intA<-function(z){ z^{(2*n-4)*exp(-alpha*z)/(z+log(x[1])-log(k))^(n-1)}}
A<-integrate(intA,lower=0,upper=Inf)$value
MSEfhx<-(n-1)*alpha^n*k^alpha/x[1]^(alpha+2)/factorial(n-2)*A-alpha^2*
k^{(2*alpha)/x[1]^{(2*alpha+2)}
Fx<-1-(k/x[1])^alpha
intB2 < -function(w) \{ w^(-n-1) * exp(w*log(k/x[1])) * exp(-alpha*n/w) \}
B2<-integrate(intB2,lower=0,upper=Inf)$value
EFtildx<-1-(alpha*n)^n/factorial(n-1)*B2
      intB3 < -function(w) \{ w^(-n-1)*exp(2*w*log(k/x[1]))*exp(-alpha*n/w) \}
      B3<-integrate(intB3,lower=0,upper=Inf)$value
      EFtildxs2 < -1-2*(alpha*n)^n/factorial(n-1)*B2+(alpha*n)^n/factorial(n-1)*B3
MSEFtildx<-EFtildxs2-2*Fx*EFtildx+Fx^2
intA1 < function(w) \{ (w-log(x[1]/k))^(2*n-2)*exp(-alpha*w)*w^(-n+1) \}
A1<-integrate(intA1,lower=log(x[1]/k),upper=Inf)$value
MSEFhx<-1-2*(k/x[1])^alpha+alpha^n/factorial(n-1)*A1-(1-(k/x[1])^alpha)^2
sfh<-sfh+MSEfhx
sft<-sft+MSEftildx
sFh<-sFh+MSEFhx
sFt<-sFt+MSEFtildx
}
mMSEfhx<-sfh/t
mMSEftildx<-sft/t
mMSEFhx<-sFh/t
mMSEFtildx<-sFt/t
return(c(mMSEfhx,mMSEftildx,mMSEFhx,mMSEFtildx))
sim(10,5,1,5,1)
sim1=function(t,k,alpha,r){
i < -seq(3,35,1)
for (j in i){
sim(t,j,k,alpha,r)}}
sim1(10,1,5,1)
```

4 Tables

In order to get the idea of efficiency between the two type of estimation i.e MLE and UMVUE. We have generated a sample of size 4(1)15(5)100 from the Pareto distribution with $\alpha=0.5(0.5)2$ and k=0.5(0.5)2. We have given Tables based on one thousand independent replication of each experiments.

Table 1. shows the bias and MSE of the estimators of the pdf and bias and MSE of the estimators of cdf are shown in Tables 2. The value in the bracket is for the MSE in each tables. From the Tables, it has been seen that MLE of pdf and cdf are more efficient than UMVUEs.

One should note that UMVUE of α is better than MLE of α .

Table 1. MSE of $\hat{f}(x)$ and $\tilde{f}(x)$ for different values of α and k respect to n

| | J (** | | | | | 1 | |
|---|----------------|---------------|----------------|--------------|----------------|--------------|--|
| n | $\alpha = 0.5$ | $\alpha = 1$ | $\alpha = 1.5$ | $\alpha = 2$ | $\alpha = 0.5$ | $\alpha = 2$ | |
| L., | k = 0.5 | k = 1 | k = 1.5 | k = 2 | k = 2 | k = 0.5 | |
| 4 | .4723690000 | .4755806957 | .476511 | .476705 | .029389 | 7.658800 | |
| | (.4551780000) | (.4572692552) | (.457605) | (.457264) | (.028230) | (7.374350) | |
| 5 | .3172310000 | .3207780187 | .322473 | .321924 | .019827 | 5.157680 | |
| | (.2725650000) | (.2757554306) | (.277487) | (.276551) | (.017036) | (4.434020) | |
| 6 | .2383780000 | .2404067792 | .241698 | .242471 | .014881 | 3.885290 | |
| | (.1881800000) | (.1896539273) | (.190811) | (.191536) | (.011743) | (3.071030) | |
| 7 | .1914040000 | .1929655509 | .193791 | .194454 | .011901 | 3.105110 | |
| | (.1421410000) | (.1432299232) | (.143884) | (.144461) | (.008828) | (2.305270) | |
| 8 | .1599510000 | .1602595912 | .161806 | .161604 | .009946 | 2.588020 | |
| | (.1133700000) | (.1133783291) | (.114652) | (.114415) | (.007042) | (1.832800) | |
| 9 | .1372420000 | .1380770369 | .138780 | .139051 | .008601 | 2.213210 | |
| | (.0937400000) | (.0942468652) | (.094764) | (.094956) | (.005877) | (1.509400) | |
| 10 | .1195130000 | .1209664617 | .121456 | .121541 | .007514 | 1.944830 | |
| | (.0791650000) | (.0801522609) | (.080488) | (.080528) | (.004982) | (1.288590) | |
| 11 | .1063690000 | .1075933345 | .107972 | .108015 | .006661 | 1.728000 | |
| | (.0687560000) | (.0695600949) | (.069808) | (.069818) | (.004307) | (1.116900) | |
| 12 | .0957350000 | .0966263202 | .096937 | .097635 | .005985 | 1.561640 | |
| 12 | (.0606160000) | (.0611705833) | (.061363) | (.061865) | (.003789) | (.989450) | |
| 13 | .0873630000 | .0876848730 | .088438 | .088472 | .005442 | 1.418000 | |
| 13 | (.0543730000) | (.0545292384) | (.055038) | (.055049) | (.003386) | (.882550) | |
| 14 | .0802730000 | .0808533231 | .081013 | .081069 | .005006 | 1.299260 | |
| 14 | | | (.049654) | (.049681) | (.003069) | (.796410) | |
| 1.5 | (.0492200000) | (.0495643965) | | | | | |
| 15 | .0735700000 | .0746156473 | .074685 | .074919 | .004613 | 1.197330 | |
| 2.0 | (.0444890000) | (.0451395324) | (.045168) | (.045318) | (.002790) | (.724140) | |
| 20 | .0532991000 | .0538278427 | .053924 | .054039 | .003330 | .865418 | |
| | (.0307735000) | (.0310780487) | (.031129) | (.031196) | (.001923) | (.499644) | |
| 25 | .0418259000 | .0420976064 | .042261 | .042273 | .002612 | .676949 | |
| | (.0234799000) | (.0236273808) | (.023720) | (.023724) | (.001466) | (.379933) | |
| 30 | .0341756000 | .0346115111 | .034740 | .034775 | .002141 | .556918 | |
| | (.0188200000) | (.0190619643) | (.019133) | (.019152) | (.001179) | (.306728) | |
| 35 | .0290451000 | .0294767054 | .029444 | .029448 | .001821 | .473159 | |
| | (.0157795000) | (.0160172951) | (.015996) | (.015996) | (.000989) | (.257090) | |
| 40 | .0252562000 | .0254701187 | .025650 | .025593 | .001581 | .411040 | |
| | (.0135817000) | (.0136957411) | (.013795) | (.013761) | (.000850) | (.221060) | |
| 45 | .0224046000 | .0225300956 | .022630 | .022658 | .001395 | .362473 | |
| | (.0119535000) | (.0120188221) | (.012073) | (.012088) | (.000744) | (.193366) | |
| 50 | .0198892000 | .0202331230 | .020334 | .020295 | .001253 | .324570 | |
| | (.0105411000) | (.0107253823) | (.010780) | (.010758) | (.000664) | (.172036) | |
| 55 | .0181373000 | .0183714478 | .018323 | .018382 | .001130 | .294111 | |
| | (.0095638000) | (.0096881397) | (.009661) | (.009692) | (.000596) | (.155077) | |
| 60 | .0165965000 | .0167227711 | .016761 | .016767 | .001036 | .269022 | |
| | (.0087132000) | (.0087790315) | (.008799) | (.008802) | (.000544) | (.141232) | |
| 65 | .0152848000 | .0153716098 | .015452 | .015489 | .000954 | .247479 | |
| | (.0079948000) | (.0080394109) | (.008082) | (.008102) | (.000499) | (.129436) | |
| 70 | .0141436000 | .0142593414 | .014292 | .014337 | .000880 | .228735 | |
| 10 | (.0073741000) | (.0074341631) | (.007451) | (.007474) | (.000459) | (.119243) | |
| 75 | .0131255000 | .0132738528 | .013345 | .013349 | .000820 | .213970 | |
| 13 | (.0068239000) | (.0069011346) | (.006938) | (.006940) | (.000427) | (.111250) | |
| 80 | .0123438000 | .0124114845 | .012482 | .012486 | .000766 | .199643 | |
| 80 | (.0064025000) | (.0064370354) | (.006474) | (.006476) | (.000397) | (.103542) | |
| 0.5 | | | | | | | |
| 85 | .0115424000 | .0117098555 | .011725 | .011738 | .000722 | .188222 | |
| | (.0059735000) | (.0060606327) | (.006068) | (.006075) | (.000374) | (.097417) | |
| 90 | .0109372000 | .0110042351 | .011088 | .011080 | .000680 | .177858 | |
| | (.0056499000) | (.0056841841) | (.005728) | (.005724) | (.000351) | (.091881) | |
| 95 | .0102967000 | .0104005835 | .010482 | .010493 | .000644 | .167683 | |
| | (.0053096000) | (.0053631148) | (.005405) | (.005411) | (.000332) | (.086471) | |
| 100 | .0098069000 | .0098648715 | .009939 | .009971 | .000615 | .159213 | |
| | (.0050495000) | (.0050790332) | (.005117) | (.005134) | (.000316) | (.081978) | |
| bracket refers to the MSE of MLE of $f(x)$ ($\tilde{f}(x)$) and without bracket refers to the MSE of IIMV | | | | | | | |

The figures in the bracket refers to the MSE of MLE of f(x) ($\tilde{f}(x)$) and without bracket refers to the MSE of UMVUE of f(x) ($\tilde{f}(x)$)

Table 2. MSE of $\hat{F}(x)$ and $\tilde{F}(x)$ for different values of α and k respect to n

| n | $\alpha = 0.5$ | $\alpha = 1$ | $\alpha = 1.5$ | $\alpha = 2$ | $\alpha = 0.5$ | $\alpha = 2$ |
|-----|----------------|---------------|----------------|--------------|----------------|--------------------|
| | k = 0.5 | k = 1 | k = 1.5 | k = 2 | k = 2 .121205 | k = 0.5 .157459 |
| 4 | .1333908446 | .1368410441 | .038435 | .115906 | | |
| | (.0014763889) | (.0012562228) | (.001100) | (.000794) | (.001880) | (.002199) |
| 5 | .1118961062 | .1799791911 | .172372 | .198062 | .137497 | .124812 |
| | (.0072949912) | (.0356532489) | (.029683) | (.055201) | (.013072) | (.009786) |
| 6 | .1408169615 | .1737093067 | .101804 | .188658 | .143735 | .134947 |
| | (.0037596610) | (.0032023158) | (.001684) | (.005213) | (.003836) | (.005169) |
| 7 | .2191326212 | .2065645792 | .195758 | .250334 | .126943 | .160016 |
| | (.0229514291) | (.0165453192) | (.013057) | (.067911) | (.004378) | (.007159) |
| 8 | .2007151499 | .2012409768 | .195431 | .219443 | .197351 | .203045 |
| | (.0034259269) | (.0032392638) | (.004168) | (.000621) | (.002333) | (.007496) |
| 9 | .2576817407 | .2512987261 | .147364 | .135785 | .124131 | .269592 |
| | (.0275929634) | (.0205299145) | (.004077) | (.003471) | (.002910) | (.060899) |
| 10 | .1492978531 | .1837933180 | .150179 | .220440 | .134771 | .213361 |
| | (.0023002929) | (.0030831665) | (.003979) | (.005504) | (.001508) | (.007332) |
| 11 | .2623413820 | .2259891179 | .259195 | .274008 | .163600 | .129209 |
| | (.0139901431) | (.0076116749) | (.012447) | (.046732) | (.003893) | (.002419) |
| 12 | .1819439058 | .1679376346 | .197488 | .176699 | .147436 | .205010 |
| | (.0062366813) | (.0049937937) | (.004988) | (.004595) | (.004263) | (.005586) |
| 13 | .2177045578 | .1674005776 | .235847 | .254445 | .141234 | .141272 |
| | (.0058142184) | (.0033358805) | (.006945) | (.008379) | (.002349) | (.002350) |
| 14 | .2356903001 | .0548103487 | .227515 | .193420 | .188948 | .136300 |
| | (.0060898969) | (.0003127263) | (.006204) | (.004149) | (.006286) | (.001994) |
| 15 | .1076037454 | .1555011068 | .122969 | .253257 | .199792 | .191199 |
| | (.0011259936) | (.0024132218) | (.001482) | (.007139) | (.004118) | (.003743) |
| 20 | .2627256864 | .2622509943 | .248371 | .232384 | .164861 | .168388 |
| | (.0065292099) | (.0062066109) | (.005120) | (.004292) | (.002324) | (.002052) |
| 25 | .2488455377 | .2598193829 | .218770 | .259375 | .151849 | .177311 |
| | (.0041453670) | (.0052022850) | (.004877) | (.004883) | (.001275) | (.001795) |
| 30 | .2251732413 | .2172670151 | .257171 | .211684 | .113000 | .118642 |
| | (.0042208658) | (.0023800197) | (.004369) | (.003983) | (.000552) | (.000612) |
| 35 | .2458125666 | .2072001232 | .198559 | .204366 | .178507 | .156018 |
| | (.0028694984) | (.0018011252) | (.003254) | (.003352) | (.001266) | (.000935) |
| 40 | .1737078950 | .2554091293 | .251794 | .152611 | .116046 | .156706 |
| | (.0010316986) | (.0032353869) | (.002759) | (.002133) | (.001487) | (.000818) |
| 45 | .1940551896 | .2429150460 | .173837 | .222621 | .157868 | .108663 |
| | (.0011808928) | (.0021535385) | (.002250) | (.002867) | (.000733) | (.000325) |
| 50 | .1064113870 | .2267825379 | .179884 | .228218 | .125222 | .151083 |
| | (.0011050152) | (.0015767910) | (.000884) | (.002635) | (.000394) | (.000594) |
| 55 | .2299332896 | .1855659085 | .252088 | .218652 | .163238 | .181364 |
| | (.0024125373) | (.0008607060) | (.002056) | (.002338) | (.000639) | (.001967) |
| 60 | .1392457773 | .1773309892 | .127104 | .227032 | .170991 | .108474 |
| L | (.0013392528) | (.0017724707) | (.001195) | (.002202) | (.000649) | (.000238) |
| 65 | .1986728058 | .2322009300 | .253793 | .115857 | .107674 | .118045 |
| | (.0018421105) | (.0012944937) | (.001897) | (.000989) | (.000215) | (.000262) |
| 70 | .2288138955 | .1496013496 | .251293 | .108626 | .134125 | .123655 |
| | (.0019021751) | (.0012705637) | (.001864) | (.000203) | (.000320) | (.000268) |
| 75 | .2524003249 | .1656385542 | .252058 | .226321 | .113991 | .124311 |
| | (.0017059572) | (.0013387601) | (.001537) | (.001769) | (.000209) | (.000252) |
| 80 | .2232959249 | .1154621504 | .238145 | .123070 | .159104 | .104494 |
| | (.0016486241) | (.0008181671) | (.001686) | (.000887) | (.000408) | (.000719) |
| 85 | .1569167081 | .1116721695 | .197303 | .038435 | .100752 | .160423 |
| | (.0003712149) | (.0007421171) | (.000637) | (.000164) | (.000141) | (.000959) |
| 90 | .2498099645 | .1551047337 | .154748 | .084016 | .080884 | .071267 |
| | (.0014551335) | (.0003408055) | (.000339) | (.000483) | (.000459) | (.000064) |
| 95 | .2525238629 | .1423488455 | .237976 | .038663 | .021378 | .055160 |
| | (.0013016097) | (.0002657662) | (.000956) | (.000151) | (.000005) | (.000255) |
| 100 | .1352354632 | .0786840925 | .105801 | .064531 | .055326 | .026498 |
| L | (.0008108742) | (.0004021173) | (.000132) | (.000305) | (.000245) | (.000081) |
| | | CMIE CEC) | ~ | | | MCE CHAI |

The figures in the bracket refers to the MSE of MLE of F(x) ($\hat{F}(x)$) and without bracket refers to the MSE of UMVUE of F(x) ($\hat{F}(x)$)

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Efficient estimation in the Pareto distribution

U.J. Dixit, M. Jabbari Nooghabi*

Department of Statistics, University of Mumbai, Mumbai, India

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ABSTRACT

The maximum likelihood estimation (MLE) of the probability density function (pdf) and cumulative distribution function (CDF) are derived for the Pareto distribution. It has been shown that MLEs are more efficient than uniform minimum variance unbiased estimators of pdf and CDF.

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1. Introduction

The Pareto distribution has been used in connection with studies of income, property values, insurance risk, migration, size of cities and firms, word frequencies, business mortality, service time in queuing systems, etc.

The objective of this paper is to discuss efficient estimation of probability density function (pdf) and cumulative distribution function (CDF) of Pareto distribution which has been one of the most distinguished candidates for the honor of explaining the distribution of incomes, assets, etc.

We assume that random variable X has Pareto distribution with parameters α and k (known) and its pdf is as

$$f(x) = \frac{\alpha k^{\alpha}}{x^{\alpha+1}}, \quad 0 < k \le x, \alpha > 0,$$

and CDF is

$$F(x) = 1 - \left(\frac{k}{x}\right)^{\alpha}, \quad k \le x.$$

E-mail addresses: ulhasdixit@yahoo.co.in (U.J. Dixit), jabbarinm@yahoo.com (M. Jabbari Nooghabi).

^{*} Corresponding author.

In economics, where this distribution is used as an income distribution, *k* is some minimum income with a known value. As abadi [1] derived the uniformly minimum variance unbiased estimator (UMVUE) of pdf, CDF and the *r*th moment.

In general, we expect that the UMVU estimators are better than MLEs. In Pareto distribution, we show that UMVU estimators of parameter α is more efficient than the MLE, but for pdf and CDF, ML estimators are biased and more efficient than the UMVUEs.

2. Maximum likelihood estimator

Let X_1, X_2, \ldots, X_n be a random sample of size n from the Pareto distribution. According to ML method we obtain the MLE of α and it is given as $\tilde{\alpha}$ where $\tilde{\alpha} = n(\sum_{i=1}^n \ln(\frac{x_i}{k}))^{-1}$.

Therefore, by using the property of MLE we can obtain the estimator of pdf and CDF with replacement of $\tilde{\alpha}$ instead of α in the pdf and CDF, respectively. Then

$$\tilde{f}(x) = \frac{\tilde{\alpha}k^{\tilde{\alpha}}}{x^{\tilde{\alpha}+1}}, \quad \tilde{\alpha} > 0, \ 0 < k \le x, \tag{1}$$

$$\tilde{F}(x) = 1 - \left(\frac{k}{x}\right)^{\tilde{\alpha}}, \quad 0 < k \le x, \tilde{\alpha} > 0.$$
 (2)

We know that pdf of $S = \sum_{i=1}^{n} \ln(\frac{X_i}{k})$ is

$$g(s) = \frac{\alpha^n s^{n-1}}{\Gamma(n)} \exp(-\alpha s), \quad s > 0,$$
(3)

and by using some elementary algebra, we can find the distribution of $w= ilde{lpha}$ as

$$g(w) = \frac{(\alpha n)^n}{\Gamma(n)w^{n+1}} \exp\left\{-\frac{\alpha n}{w}\right\}, \quad w > 0.$$
 (4)

Note. It is clear that the MLE of α is biased and $MSE(\tilde{\alpha}) = \frac{\alpha^2(n^2+n-2)}{(n-1)^2(n-2)}$

Theorem 1. (A) $\tilde{f}(x)$ is a biased estimator of f(x) and

$$E(\tilde{f}(x)) = \frac{1}{\Gamma(n)x} \sum_{i=0}^{n-2} \frac{(\alpha n)^{j+1}}{j!} \Gamma(n-j-1) \left(\ln\left(\frac{k}{x}\right) \right)^{j}.$$
 (5)

(B) $\tilde{F}(x)$ is a biased estimator of F(x) and

$$E(\tilde{F}(x)) = 1 - \frac{1}{\Gamma(n)} \sum_{i=0}^{n-1} \frac{(\alpha n)^j}{j!} \Gamma(n-j) \left(\ln\left(\frac{k}{x}\right) \right)^j.$$
 (6)

Proof. In cases of (A), we can easily find the expectation of f(x) with substituting this formula: $(\frac{k}{x})^w = e^{w \ln(\frac{k}{x})} = \sum_{j=0}^{\infty} \frac{w^j (\ln(\frac{k}{x}))^j}{j!}$. Also, the GAMMA function defines for variable grater than zero, then j must be less than (n-1) and the proof is complete. In the case (B), the proof is similar as in the case (A). \square

Theorem 2.

(A)

$$\begin{aligned} \text{MSE}(\tilde{f}(x)) &= \frac{1}{\Gamma(n)x^2} \sum_{j=0}^{n-3} \frac{2^j (\alpha n)^{j+2}}{j!} \Gamma(n-j-2) \left(\ln \left(\frac{k}{x} \right) \right)^j \\ &- 2 \frac{\alpha k^{\alpha}}{\Gamma(n)x^{\alpha+2}} \sum_{j=0}^{n-2} \frac{(\alpha n)^{j+1}}{j!7} \Gamma(n-j-1) \left(\ln \left(\frac{k}{x} \right) \right)^j + \left(\frac{\alpha k^{\alpha}}{x^{\alpha+1}} \right)^2. \end{aligned} \tag{7}$$

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(B)

$$MSE(\tilde{F}(x)) = \frac{1}{\Gamma(n)} \sum_{j=0}^{n-1} \frac{2^{j} (\alpha n)^{j}}{j!} \Gamma(n-j) \left(\ln \left(\frac{k}{x} \right) \right)^{j}$$
$$-2 \left(\frac{k}{x} \right)^{\alpha} \frac{1}{\Gamma(n)} \sum_{j=0}^{n-1} \frac{(\alpha n)^{j}}{j!} \Gamma(n-j) \left(\ln \left(\frac{k}{x} \right) \right)^{j} + \left(\frac{k}{x} \right)^{2\alpha}. \tag{8}$$

Proof. In cases (A) and (B) we should find $E(\tilde{f}(x))^2$ and $E(\tilde{F}(x))^2$ as the previous Theorem, respectively. So by using some elementary algebra the proof is complete. \Box

3. MSE of UMVU estimator

Asrabadi [1] derived the UMVUE of α , f(x) and F(x). Here UMVUE of α , f(x) and F(x) are denoted by $\hat{\alpha}$, $\hat{f}(x)$ and $\hat{F}(x)$, respectively. So

$$\hat{\alpha} = \frac{n-1}{\ln(t) - n \ln(k)},\tag{9}$$

$$\hat{f}(x) = \frac{(n-1)[\ln(t) - \ln(x) - (n-1)\ln(k)]^{n-2}}{x[\ln(t) - n\ln(k)]^{n-1}},\tag{10}$$

and

$$\hat{F}(x) = 1 - \frac{\left[\ln(t) - \ln(x) - (n-1)\ln(k)\right]^{n-1}}{\left[\ln(t) - n\ln(k)\right]^{n-1}},\tag{11}$$

where $k \le x \le tk^{1-n}$, and $t = \prod_{i=1}^n x_i$ is the observed value of T.

Theorem 3.

(A)

$$MSE(\hat{f}(x)) = \frac{(n-1)\alpha^{2}k^{\alpha}}{\Gamma(n-1)x^{\alpha+2}} \sum_{j=0}^{n-3} C(2n-4,j)\alpha^{j}\Gamma(n-j-2) \left(-\ln\left(\frac{x}{k}\right)\right)^{j}$$

$$\times \sum_{i=0}^{n-3-j} \frac{\alpha^{i} \left(\ln(\frac{x}{k})\right)^{i}}{i!} - \left(\frac{\alpha k^{\alpha}}{x^{\alpha+1}}\right)^{2}, \qquad (12)$$

(B)

$$MSE(\hat{F}(x)) = \frac{k^{\alpha}}{\Gamma(n)x^{\alpha}} \sum_{j=0}^{n-1} C(2n-2, j)\alpha^{j} \Gamma(n-j) \left(-\ln\left(\frac{x}{k}\right)\right)^{j}$$

$$\times \sum_{i=0}^{n-j-1} \frac{\alpha^{i} \left(\ln(\frac{x}{k})\right)^{i}}{i!} - \left(\frac{k}{x}\right)^{2\alpha}, \qquad (13)$$

where $C(n, k) = \frac{n!}{k!(n-k)!}$

Proof. In cases (A) and (B), we can obtain $E(\hat{f}(x))^2$ and $E(\hat{F}(x))^2$ by using pdf of T that is given in [1]. In the process to calculate the integral we should note that

$$\int_{k}^{\infty} \frac{z^{n-1} \alpha^{n}}{\Gamma(n)} e^{-\alpha z} = \sum_{i=0}^{n-1} \frac{(\alpha k)^{i}}{i!} e^{-\alpha k}.$$

Hence, the proof is complete. \Box

Note. One should note that $\text{MSE}(\hat{\alpha}) = \frac{\alpha^2}{(n-2)}$. ¹⁸

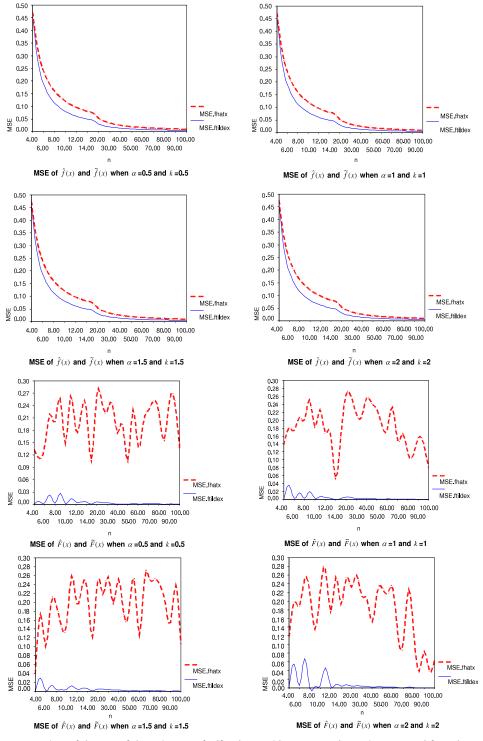


Fig. 1. Comparison of the MSE of the estimators of pdf and CDF with respect to observation generated from the Pareto distribution. 19

4. Comparison of MLE and UMVUE

It is obvious that the UMVU estimator of α is more efficient than the MLE for any value of n. Now in order to get the idea of efficiency between MLE and UMVUE of pdf and CDF, we have generated a sample of size 4(1)15(5)100 from the Pareto distribution with $\alpha=0.5(0.5)2$ and k=0.5(0.5)2. We have given graphs based on one thousand independent replications of each experiments (Fig. 1). From the graphs, it has been seen that MLE of pdf and CDF are more efficient than UMVUEs.

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On estimation for the Pareto distribution



Hui He, Na Zhou, Ruiming Zhang*

College of Science, Northwest A&F University, Yangling, Shaanxi 712100, PR China

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ABSTRACT

In this work, we obtain the r-th raw moments of the probability density function (PDF) and reliability function (RF) for the Pareto distribution under the maximum likelihood estimation (MLE) and uniform minimum variance unbiased estimation (UMVUE). We derive some large sample properties of the estimators, the MLE and UMVUE of the PDF as well as RF. Two examples are provided to compute the efficient estimations of PDF and RF numerically. Our results indicate that there are no absolute superiorities of MLEs over the UMVUEs of PDF and RF and vice versa.

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1. Introduction

We consider a random variable *X* has the Pareto distribution (PD) with PDF

$$f(x) = \frac{\alpha k^{\alpha}}{x^{\alpha+1}},\tag{1}$$

and RF

$$G(x) = \text{Prob}\{X > x\} = \left(\frac{k}{x}\right)^{\alpha},\tag{2}$$

where α is a shape parameter ($\alpha > 0$), and k is a scale parameter (known, and x > k > 0). In economics, k usually represents some minimum income with a known value, see [1].

PD was applied by Pareto [7] to model the allocation of wealth among individuals and the distribution of incomes. It has been widely used in economics, insurance (general liability, commercial

^{*} Corresponding author. Tel.: +86 13032906582.

E-mail addresses: hhznby@163.com (H. He), zhouna871208@163.com (N. Zhou), ruiming_nwsuaf@163.com, ruimingzhang@yahoo.com (R. Zhang).

auto [9]), geography (sizes of human settlements [8]), physical sciences (sizes of sand particles or meteorites [8], clusters of Bose–Einstein condensate near absolute zero [5]), chemical sciences (distributions of electrolytic powder production [4]). Asrabadi [1] established the UMVUEs for the PDF and cumulative distribution function (CDF) of PD. Based on the work of Asrabadi [1], Dixit and Jabbari Nooghabi [2] tried to study the mean square errors (MSEs) of the MLEs and UMVUEs for the PDF and CDF of PD and their results seem to show that the MLEs are more efficient than the UMVUEs of PDF and CDF. Unfortunately, their work are seriously flawed. Most of their main claims in [2] are wrong, and their conclusion, the MLEs are more efficient than the UMVUEs of PDF and CDF, is unreasonable.

We present our main results in Section 2. Most of the results in Section 2.1 are corrected versions of the wrong results of [2]. We also notice that the exact expressions of the MSEs of estimators of PDF and RF may not be useful in case of large scale samples and large scale numerical computations. For this reason we have derived the asymptotic expressions of the r-th raw moments and MSEs in Section 2.2. Two numerical examples are provided in Section 2.3 to show how to compute the efficient estimations of PDF and RF. In Section 2.4 we expose the fatal errors in [2].

2. Main results

As a notational convenience, let

$$z = z(x) = \log \frac{x}{k}$$
, $z_x = z_x(x) = \frac{dz}{dx} = \frac{1}{x}$,

through the rest of this paper. It is known that the UMVUEs of f(x) and G(x) are given by [1]

$$\hat{f}(x) = \frac{n-1}{s} z_x \left(1 - \frac{z}{s}\right)^{n-2}, \qquad \hat{G}(x) = \left(1 - \frac{z}{s}\right)^{n-1},$$

where z < s, $s = \sum_{i=1}^{n} z(x_i)$ and s follow the Gamma distribution $Ga(n, \alpha)$. Note that the UMVUE of α is $\hat{\alpha} = (n-1)/s$.

The MLEs of f(x) and G(x) can be computed easily, they are

$$\tilde{f}(x) = \tilde{\alpha} z_x e^{-\tilde{\alpha} z}, \qquad \tilde{G}(x) = e^{-\tilde{\alpha} z},$$

where $\tilde{\alpha} = \frac{n}{s}$ is the MLE of α . Note that the PDF of s is given by $h(s) = \frac{\alpha^n s^{n-1}}{\Gamma(n)} \exp(-\alpha s)$.

2.1. The r-th raw moments of estimations

Theorem 1. For n > r > 0, the r-th raw moments of $\tilde{f}(x)$ and $\tilde{G}(x)$ are given by

$$E(\tilde{f}(x))^{r} = \frac{2}{\Gamma(n)} (n\alpha z_{x})^{r} (\sqrt{nr\alpha z})^{n-r} K_{n-r} (2\sqrt{nr\alpha z}),$$
(3)

$$E(\tilde{G}(x))^{r} = \frac{2}{\Gamma(n)} (\sqrt{nr\alpha z})^{n} K_{n} \left(2\sqrt{nr\alpha z}\right), \tag{4}$$

where $K_{\nu}(x)$ is the modified Bessel function [6].

Proof. For the proof we just need to note the well-known integral representation [6],

$$K_{\nu}(x) = \frac{1}{2} \left(\frac{x}{2}\right)^{\nu} \int_{0}^{\infty} \exp\left(-t - \frac{x^{2}}{4t}\right) \frac{dt}{t^{\nu+1}}. \quad \Box$$

Corollary 1. The mean square errors of $\tilde{f}(x)$ and $\tilde{G}(x)$ are given by

$$MSE(\tilde{f}(x)) = \frac{2(n\alpha z_{x})^{2}}{\Gamma(n)} (\sqrt{2n\alpha z})^{n-2} K_{n-2} \left(2\sqrt{2n\alpha z}\right) - \frac{4n\alpha z_{x}}{\Gamma(n)} f(x) (\sqrt{n\alpha z})^{n-1} K_{n-1}^{22} \left(2\sqrt{n\alpha z}\right) + f^{2}(x).$$

$$(5)$$

$$MSE(\tilde{G}(x)) = \frac{2}{\Gamma(n)} (\sqrt{2n\alpha z})^n K_n \left(2\sqrt{2n\alpha z}\right) - \frac{4}{\Gamma(n)} G(x) (\sqrt{n\alpha z})^n K_n \left(2\sqrt{n\alpha z}\right) + G^2(x).$$
(6)

Theorem 2. For n > r > 1, the r-th raw moments of $\hat{f}(x)$ and $\hat{G}(x)$ are given by

$$E(\hat{f}(x))^{r} = (\alpha z_{x}(n-1))^{r-1} f(x) \frac{\Gamma(nr-2r+1)}{\Gamma(n-1)} U(nr-n-r+1, r-n+1, \alpha z), \tag{7}$$

$$E(\hat{G}(x))^{r} = \frac{\Gamma(nr - r + 1)}{\Gamma(n)}G(x)U(nr - n - r + 1, 1 - n, \alpha z),$$
(8)

where U(a, b, c) is the Kummer confluent hypergeometric function [6].

Proof. Note that the Kummer confluent hypergeometric function has an integral representation [6],

$$U(a, b, c) = \frac{e^{c}}{\Gamma(a)} \int_{0}^{\infty} t^{a-1} (1+t)^{b-a-1} e^{-ct} dt,$$

and the proof is completed by applying the Kummer transformation [6],

$$U(a, b, c) = c^{1-b}U(1 + a - b, 2 - b, c).$$

Corollary 2. The mean square errors of $\hat{f}(x)$ and $\hat{G}(x)$ are given by

$$MSE(\hat{f}(x)) = \alpha z_x f(x) (n-1) \frac{\Gamma(2n-3)}{\Gamma(n-1)} U(n-1, 3-n, \alpha z) - f^2(x).$$
 (9)

$$MSE(\hat{G}(x)) = G(x) \frac{\Gamma(2n-1)}{\Gamma(n)} U(n-1, 1-n, \alpha z) - G^{2}(x).$$
(10)

Figs. 1 and 2 illustrate the efficient estimators between MLEs and UMVUEs of PDF and RF for $n = \{5, 6, 30\}$, $k = \{1, 5\}$, $0 < \alpha < 10$ and k < x < 50. In the graphs, the black areas indicate that the MLEs of PDF and RF are more efficient than the UMVUEs while the white areas mean the UMVUEs of PDF and RF are more efficient than the MLEs, and there is no evidence that the black areas or the white areas will disappear from the first quadrant. Thus we conclude that the MLEs are not generally more efficient than the UMVUEs of PDF/RF and vice versa. We also notice that Corollaries 1 and 2 can help us to obtain more efficient estimations, see Example 1.

2.2. The convergence rate of estimators

Corollaries 1 and 2 can be expediently used to calculate the MSEs of the estimations for a small sample. However, in practice, we find that the corollaries would not be expedient for a large sample and the large-scale numerical computation. For reasonable large n, direct numerical evaluations of $\Gamma(2n-1)$, $K_n\left(2\sqrt{n\alpha z}\right)$, $U(n-1,1-n,\alpha z)$ will incur either overflow or underflow. For large-scale numerical computation, symbolic computations will run for a very long time. Therefore, it is necessary to study the large sample properties of the r-th raw moments and the asymptotic behaviors of the MSEs.

Lemma 1. For fixed y > 0, $a \ge 0$, then

$$K_{\nu}\left(2\sqrt{y(\nu+a)}\right) = e^{-\nu-a-y}\sqrt{\pi(\nu+\frac{23}{4})^{\nu}(2\nu y^{\nu})^{-1}}\sum_{i=0}^{\infty}\frac{a_{i}(a,y)}{\nu^{i}},$$
(11)

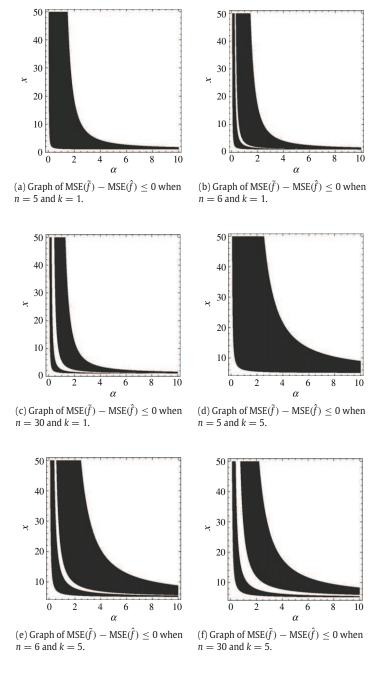


Fig. 1. Efficient estimators between MLE and UMVUE of PDF.

as $\nu \to +\infty$, where the first three coefficients are given by

$$\begin{split} a_0(a,y) &= 1, \qquad a_1(a,y) = y^2/2 - (a+1)\,y + (6a^2+1)/12, \\ a_2(a,y) &= y^4/8 - (3a+7)y^3/6 + \left(18a^2+48a+61\right)y^2/24 \\ &- \left(144a^3+144a^2+312a+312\right)y/288 + \left(36a^4-96a^3+12a^2+1\right)/288. \end{split}$$

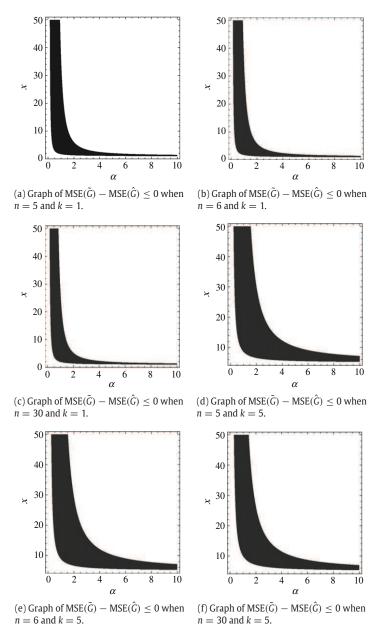


Fig. 2. Efficient estimators between MLE and UMVUE of RF.

Lemma 2. For fixed y > 0, r > 1, $+\infty > a > -\infty$, then

$$\Gamma(nr - n - r + 1)U(nr - n - r + 1, ar - r - n + 1, y)$$

$$= (r - 1)^{nr - n - r + \frac{1}{2}} e^{y - ry} r^{ar - nr - \frac{1}{2}} \sqrt{2\pi n^{-1}} \sum_{i=0}^{\infty} \frac{b_i(a, r, y)}{n^i},$$
(12)

as $n \to +\infty$, where the first coefficient is given as $b_0(a,r,y)=1$, and when r=2, the second and third coefficients are given by

$$b_1(a, 2, y) = y^2 + 2(1 - a)y + (8a^2 - 20a + 17)/8,$$

$$b_2(a, 2, y) = y^4/2 - 2ay^3 + (384a^2 - 192a + 144)y^2/128 + (-256a^3 + 384a^2 - 416a + 288)y/128 + (64a^4 - 192a^3 + 320a^2 - 408a + 289)/128.$$

Note that the proofs of Lemmas 1 and 2 are very lengthy, so we only provide the results here. For interested readers, please see the related sections on Laplace's method in [10].

Theorem 3. If $n \to \infty$, then the following formulas hold:

- (a) $E(\tilde{f}(x))^r = f^r(x) + \mathcal{O}(n^{-1}).$
- (b) $E(\tilde{G}(x))^r = G^r(x) + \mathcal{O}(n^{-1}).$
- (c) $E(\hat{f}(x))^r = f^r(x) + \mathcal{O}(n^{-1}).$
- (d) $E(\hat{G}(x))^r = G^r(x) + \mathcal{O}(n^{-1}).$

Proof. Cases (a) and (b) are obtained by applying Lemma 1 and Stirling's formula [6], the proofs for the cases (c) and (d) are similar to the cases (a) and (b) except applying Lemma 2 instead of Lemma 1.

Theorem 4. For $n \to \infty$, we have the following results:

(a)
$$MSE(\tilde{f}(x)) = f^2(x) ((\alpha z - 1)^2 n^{-1} + p_1(\alpha z) n^{-2} + \mathcal{O}(n^{-3})),$$

(b)
$$MSE(\tilde{G}(x)) = G^2(x) ((\alpha z)^2 n^{-1} + q_1(\alpha z) n^{-2} + \mathcal{O}(n^{-3})),$$

(c)
$$MSE(\hat{f}(x)) = f^2(x) ((\alpha z - 1)^2 n^{-1} + p_2(\alpha z) n^{-2} + \mathcal{O}(n^{-3})),$$

(d)
$$MSE(\hat{G}(x)) = G^2(x) ((\alpha z)^2 n^{-1} + q_2(\alpha z) n^{-2} + \mathcal{O}(n^{-3})),$$

where the functions p_1 , p_2 , q_1 , q_2 are defined as

$$p_1(x) = 7x^4/4 - 14x^3 + 32x^2 - 24x + 5,$$
 $q_1(x) = 7x^4/4 - 7x^3 + 5x^2,$ $p_2(x) = x^4/2 - 4x^3 + 10x^2 - 8x + 2,$ $q_2(x) = x^4/2 - 2x^3 + 2x^2,$ $x > 0.$

Proof. The proof of Theorem 4 is similar to the proof of Theorem 3. \Box

Next, we will discuss the efficient estimations under the large scale sample. By Theorem 4, we have

$$MSE(\tilde{f}(x)) - MSE(\hat{f}(x)) \approx (f(x)/n)^2 p(\alpha z),$$

and

$$MSE(\tilde{G}(x)) - MSE(\hat{G}(x)) \approx (G(x)/n)^2 q(\alpha z),$$

where $p(\alpha z) = p_1(\alpha z) - p_2(\alpha z)$ and $q(\alpha z) = q_1(\alpha z) - q_2(\alpha z)$. Note that $p(\alpha z) = 5(\alpha z)^4/4 - 10(\alpha z)^3 + 22(\alpha z)^2 - 16(\alpha z) + 3$, $q(\alpha z) = 5(\alpha z)^4/4 - 5(\alpha z)^3 + 3(\alpha z)^2$. We notice that the algebraic signs of $p(\alpha z)$ and $q(\alpha z)$ can approximatively determine the efficient estimators of PDF and RF when n is large.

Corollary 3. For sufficiently large n, we have the following results:

- 1. If $\alpha z \in [0.2856, 0.9168] \cup [1.857, 4.9439]$, then $p(\alpha z) \leq 0$ and the MLE is more efficient than the UMVUE of PDF.
- 2. If $\alpha z \in (0, 0.2856) \cup (0.9168, 1.8537) \cup (4.9439, \infty)$, then $p(\alpha z) > 0$ and the UMVUE is more efficient than the MLE of PDF.
- 3. If $\alpha z \in [0.7351, 3.2649]$, then $q(\alpha z) \leq 0$ and the MLE is more efficient than the UMVUE of RF.
- 4. If $\alpha z \in (0, 0.7351) \cup (3.2649, \infty)$, then $q(\alpha y) > 0$ and the UMVUE is more efficient than the MLE of RF.

2.3. Numerical example

Example 1. Efficient estimator in the small sample.

We use Dyer [3] annual wage data (in multiples of 10,000 US dollars) to illustrate our results. The values of the data are given below:

Here we suppose that the minimum wage is 10,000 US dollars. Then the pertinent data are n=30, k=1 and $\alpha\approx 5.4025$ (the UMVUE of α).

Further, let

$$MSE(\tilde{f}(x)) - MSE(\hat{f}(x)) = 0,$$

then, the positive roots are

$$x_1 = 1.0512$$
, $x_2 = 1.1794$, $x_3 = 1.3825$, $x_4 = 2.5563$.

Therefore, if $x \in [1.0512, 1.1794] \cup [1.3825, 2.5563]$, $\tilde{f}(x)$ is more efficient than $\hat{f}(x)$; if $x \in (1, 1.0512) \cup (1.1794, 1.3825) \cup (2.5563, \infty)$, $\hat{f}(x)$ is more efficient than $\tilde{f}(x)$. Similarly, let

$$MSE(\tilde{G}(x)) - MSE(\hat{G}(x)) = 0$$
,

then, the positive roots are

$$x_1 = 1.1384, \quad x_2 = 1.8443.$$

Hence, when $x \in [1.1384, 1.8443]$, $\tilde{G}(x)$ is more efficient than $\hat{G}(x)$; when $x \in (1, 1.1384) \cup (1.8443, \infty)$, $\hat{G}(x)$ is more efficient than $\tilde{G}(x)$.

Example 2. Efficient estimator in the large sample.

To compare with our first example, we let k=1 and $\alpha=5.4025$ (the values of k and α are the same as in the first example) in Corollary 3 to see that, when $x\in[1.0543,\,1.1850]\cup[1.4093,\,2.4970]$, $\tilde{f}(x)$ is more efficient than $\hat{f}(x)$; when $x\in(1,\,1.0543)\cup(1.1850,\,1.4093)\cup(2.4970,\,\infty)$, $\hat{f}(x)$ is more efficient than $\tilde{f}(x)$; if $x\in[1.1458,\,1.8300]$, $\tilde{G}(x)$ is more efficient than $\hat{G}(x)$; if $x\in(1,\,1.1458)\cup(1.8300,\,\infty)$, $\hat{G}(x)$ is more efficient than $\tilde{G}(x)$.

2.4. Some comments on [2]

The errors of the main results of [2] can be seen clearly from the following simple numerical calculation. Let $n=\{5,6,30\},\ k=\{1,5\},\ \alpha=\{1,5\},\ \alpha=\{1,5\},\ and\ x=\{2,4,6,8\},$ the mathematical expectation values and MSE values are listed in Table 1, and some of them are negative which is clearly absurd. It is not hard to see Theorems 1–3 of [2] are all wrong, where their Theorem 1 is about the mathematical expectation expressions of $\tilde{f}(x)$, $\tilde{F}(x)$; their Theorem 2 is about the MSE expressions of $\tilde{f}(x)$, $\tilde{F}(x)$, and their Theorem 3 is on the MSE expressions of $\hat{f}(x)$, $\hat{F}(x)$. Furthermore, all of our numerical simulations (Figs. 1, 2, Examples 1 and 2) also show that the main conclusion of [2], the MLEs are more efficient than the UMVUEs of PDF and CDF, is false.

3. Conclusion

We have studied the efficient estimation in PD in our work. Our results show that the efficient estimations of PDF and RF of PD depend on for variables (n, k, α, x) . Let $g_1(x), \ldots, g_m(x)$ denote the different estimations (MLE, UMVUE, Bayesian estimation, etc.) of PDF or RF, we construct the following

Table 1 Numerical values of $E(\tilde{f}(x))$, $E(\tilde{f}(x))$, $MSE(\tilde{f}(x))$, $MSE(\tilde{f}(x))$, $MSE(\hat{f}(x))$ and $MSE(\hat{f}(x))$, for $n = \{5, 6, 30\}$, $k = \{1, 5\}$, $\alpha = \{1, 5\}$ and $x = \{2, 4, 6, 8\}$. Note that the values are calculated by using the results of Dixit and Jabbari Nooghabi [2].

| (n, k, α, x) | $E(\tilde{f})$ | $E(\tilde{F})$ | $MSE(\tilde{f})$ | $MSE(\tilde{F})$ | $MSE(\hat{f})$ | $MSE(\hat{F})$ |
|---------------------|----------------|----------------|------------------|------------------|----------------|----------------|
| (5, 1, 5, 2) | -3.88e+2 | -1.29e+2 | 3.76e+3 | 2.25e+3 | 1.41e+1 | 5.73e+0 |
| (6, 1, 5, 2) | 8.28e+2 | 2.23e+2 | -2.11e+4 | -7.75e+3 | -5.31e+1 | -1.43e+1 |
| (30, 1, 5, 2) | 8.51e-2 | 9.67e-1 | -2.01e+7 | -1.89e + 5 | -2.37e+1 | -2.11e-1 |
| (5, 1, 5, 4) | -1.67e+3 | -2.26e+3 | 3.80e+3 | 3.79e + 4 | 4.63e-1 | 3.05e+0 |
| (6, 1, 5, 4) | 7.14e + 3 | 7.77e+3 | -4.34e+4 | -2.61e+5 | -3.50e+0 | -1.53e+1 |
| (30, 1, 5, 4) | 9.91e+5 | 1.89e+5 | -7.1e+14 | -1.1e+14 | -2.62e+7 | -3.76e+6 |
| (5, 5, 1, 6) | 1.55e-1 | 1.97e-1 | 2.94e-2 | 1.65e-2 | 5.45e-3 | 7.17e-3 |
| (6, 5, 1, 6) | 1.54e-1 | 1.92e-1 | -4.54e-3 | 5.45e-3 | 2.23e-3 | 5.14e-3 |
| (30, 5, 1, 6) | 1.42e-1 | 1.71e-1 | 4.64e-4 | 8.71e-4 | 4.49e-4 | 8.14e-4 |
| (5, 5, 1, 8) | 4.94e-2 | 3.95e-1 | 1.34e-1 | 5.05e-1 | 9.06e-3 | 4.61e-2 |
| (6, 5, 1, 8) | 8.71e-2 | 4.18e-1 | -9.79e-2 | -2.13e-1 | -4.11e-3 | 9.06e-3 |
| (30, 5, 1, 8) | 7.85e-2 | 3.83e-1 | 5.29e-5 | 3.07e-3 | 5.76e-5 | 2.99e-3 |

estimation

$$\bar{g}(x) = \begin{cases} g_1(x), & \text{if } x \in \left\{ x \middle| \bigcap_{i=1}^m \mathsf{MSE}(g_1(x)) \le \mathsf{MSE}(g_i(x)) \right\} \\ \cdots, & \cdots \\ g_m(x), & \text{if } x \in \left\{ x \middle| \bigcap_{i=1}^m \mathsf{MSE}(g_m(x)) \le \mathsf{MSE}(g_i(x)) \right\}, \end{cases}$$

where the parameters n, k, α are given. As an estimator of PDF or RF, it is more efficient than all of $g_i(x)$, i = 1, ..., m. It is also clear that $\bar{g}(x)$ may have discontinuities.

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Appendix. Proofs associated with Theorems 1-4 and Lemmas 1-2

Proof of Theorem 1.

$$\begin{split} E(\tilde{f}(x))^r &= \int_0^\infty (\tilde{f}(x))^r h(s) ds = \int_0^\infty \left(\frac{n}{s} z_x e^{-\frac{n}{s}z}\right)^r \alpha^n s^{n-1} \frac{\exp(-\alpha s)}{\Gamma(n)} ds \\ &= \frac{(\alpha n z_x)^r}{\Gamma(n)} \int_0^\infty (\alpha s)^{n-r-1} \exp\left(-\alpha s - \frac{\alpha n z r}{\alpha s}\right) d(\alpha s) \\ &= \frac{(\alpha n z_x)^r}{\Gamma(n)} \int_0^\infty t^{n-r-1} \exp\left(-t - \frac{(2\sqrt{\alpha n z r})^2}{4t}\right) dt \\ &= \frac{2}{\Gamma(n)} (n\alpha z_x)^r (\sqrt{n r \alpha z})^{n-r} K_{n-r} \left(2\sqrt{n r \alpha z}\right). \end{split}$$

Eq. (4) can be proved similarly, and the proof of Theorem 1 is complete.

Proof of Theorem 2.

$$E(\hat{f}(x))^{r} = \int_{z}^{\infty} (\hat{f}(x))^{r} h(s) ds = \int_{z}^{\infty} \left(\frac{n-1}{s} z_{x} \left(1 - \frac{z}{s}\right)^{n-2}\right)^{r} \alpha^{n} s^{n-1} \frac{\exp(-\alpha s)}{\Gamma(n)} ds$$
$$= \frac{(\alpha(n-1)z_{x})^{r}}{\Gamma(n)} \int_{z}^{\infty} (\alpha s)^{n-r-2} \left(1 - \frac{\alpha z}{\alpha s}\right)^{(n-2)r} \exp(-\alpha s) d(\alpha s)$$

$$= \frac{(\alpha(n-1)z_{x})^{r}}{\Gamma(n)} \int_{\alpha z}^{\infty} t^{(n-1)(1-r)} (t-\alpha z)^{(n-2)r} \exp(-t) dt$$

$$= \frac{(\alpha(n-1)z_{x})^{r}}{\Gamma(n)e^{\alpha z}} \int_{0}^{\infty} (t+\alpha z)^{(n-1)(1-r)} t^{(n-2)r} \exp(-t) dt$$

$$= \frac{(\alpha(n-1)z_{x})^{r}}{\Gamma(n)e^{\alpha z}} (\alpha z)^{n-r} \int_{0}^{\infty} (1+t)^{(n-1)(1-r)} t^{(n-2)r} \exp(-\alpha zt) dt$$

$$= \frac{(\alpha(n-1)z_{x})^{r}}{\Gamma(n)e^{\alpha z}} (\alpha z)^{n-r} \Gamma(nr-2r+1) U(nr-2r+1, n-r+1, \alpha z)$$

$$= (\alpha z_{x}(n-1))^{r-1} f(x) \frac{\Gamma(nr-2r+1)}{\Gamma(n-1)} U(nr-n-r+1, r-n+1, \alpha z).$$

Eq. (8) can be proved similarly.

Proof of Lemma 1.

$$K_{v}\left(2\sqrt{y(v+a)}\right) = \frac{(y(v+a))^{\frac{v}{2}}}{2} \int_{0}^{\infty} t^{-v-1} \exp\left(-t - \frac{y(v+a)}{t}\right) dt$$
$$= \frac{(y(v+a))^{\frac{v}{2}}}{2} \int_{0}^{\infty} \exp\left(-t - \frac{ya}{t}\right) \exp\left(-v\left(\log t + \frac{y}{t}\right)\right) \frac{dt}{t},$$

where y > 0, $a \ge 0$ and v > 0. Let $f(t) = \log t + y/t$, then

$$f'(t) = t - y/t^2$$
, $f''(t) = -t - 2y/t^3$,

it is clear f(t) has a unique minimum $1 + \log y$ at t = y. Then, we obtain Eq. (11) by Laplace's method [10]. Then the proof of Lemma 1 is complete.

Proof of Lemma 2.

$$\Gamma(nr - n - r + 1)U(nr - n - r + 1, ar - r - n + 1, y)$$

$$= \int_0^\infty t^{nr - n - r} (1 + t)^{ar - nr - 1} e^{-yt} dt$$

$$= \int_0^\infty t^{n(r - 1)} (1 + t)^{-nr} \frac{(1 + t)^{ar - 1}}{t^r} e^{-yt} dt$$

$$= \int_0^\infty e^{-n((1 - r)\log(t) + r\log(1 + t))} \frac{(1 + t)^{ar - 1}}{t^r} e^{-yt} dt$$

where $y > 0, \ r > 1, \ +\infty > a > -\infty$. Let $f(t) = (1 - r) \log(t) + r \log(1 + t)$, then

$$f'(t) = r(1+t)^{-1} - (r-1)t^{-1}$$
.

It is clear f(t) has a unique minimum $r \log r - (r-1) \log(r-1)$ at t=r-1. Then, we obtain Eq. (12) by Laplace's method [10]. Then the proof of Lemma 2 is complete.

Proof of Theorem 3. In the cases (a) and (b), the results can be easily obtained by applying Lemma 1 and Stirling's formula

$$\Gamma(n) = \sqrt{2\pi} n^{n-\frac{1}{2}} e^{-n} \sum_{k=0}^{\infty} \frac{c_i}{n^i}, \qquad c_0 = 1, \qquad c_1 = \frac{1}{12}, \qquad c_2 = \frac{1}{288}.$$

In the cases (c) and (d), the results can be derived by using Lemma 2 and Stirling's formula.

Proof of Theorem 4. The proof process of Theorem 4 is similar to the proof of Theorem 3.

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